Week 15 12,24

①

行列式 (Determinant)

從排列的概念說起：

Definition A permutation of a set S is a 1-1 onto mapping from S onto S.

(*) 廣義而言，S 不一定要是有限集合。

(**) 以下考慮的 S 都是有限集合。

For convenience, we let $S = \{1, 2, \cdots, n\}$.
$= [1, n]$

Then we have

Proposition 1 There are $n!$ permutations from $[1, n]$ onto $[1, n]$.

Notations

① A permutation α can be expressed as two arrays:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \cdots & \alpha(n) \end{pmatrix}.$$

② A permutation α can be written as the product of disjoint "cycles".

e.g. 1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$ is a permutation from $[1,6]$ onto $[1,6]$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix} = (1\,2\,3\,4)(5\,6).$$

Note here that $(1\,2\,3\,4)$ and $(5\,6)$ are also permutations! (~~of~~ $[1,6]$)

$$(1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix} \text{ and}$$

$$(5\,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix}.$$

<u>Definition</u> (Transposition)

A transposition is a 2-cycle.

e.g. 2. $(5\,6)$ is a transposition of $[1,6]$.

<u>Proposition 2</u> Any cycle can be written as a product of transpositions and the number of transpositions is either even or odd but not both.

e.g. 3 $(1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$

(想成一个函数是由三个函数所合成!).

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

More precisely, an even (odd) cycle contains an odd (even) number of transpositions.

Theorem 3 A permutation can be expressed as a product of transpositions. If the number of transpositions is even (odd), then it is an even (odd) permutation.

e.g. 4 $(1\,2\,3\,4)$ is an odd permutation ~~of~~ $[1,4]$.

$(1\,2)(3\,4)$ is an even permutation of $[1,4]$.

Theorem 4 Let $S=[1,n]$. Then there are $n!$ permutations of S, half of them are even (odd respectively

e.g. 5 $S=[1,3]$.

Odd permutations : $(1\,2), (1\,3), (2\,3)$.

Even permutations : $(1\,2\,3), (1\,3\,2), e$ (identity) e.g. 6 $S=[1,4]$

Odd permutations: $(12), (13), (14), (23), (24), (34),$
$(1234), (1324), (1423), (1243)$
$(1342), (1432)$.

Even permutations: $(12)(34), (13)(24), (14)(23),$
$(123), (132), (134), (143), (234), (243),$
$(124), (142), e$.

Notation (O_n, E_n) (Odd, Even)

We use S_n to denote the set of all permutations in $[1,n]$. Then $|S_n| = n!$, $|O_n| = n!/2$, $|E_n| = n!/2$.

Definition (Determinant)

Let A be an $n\times n$ matrix, $A=[a_{ij}]$.

Then $\underline{\det A} = \sum_{\alpha\in S_n} \delta_\alpha a_{1\alpha(1)} a_{2\alpha(2)} \cdots a_{n\alpha(n)}$ where

$$\delta_\alpha = \begin{cases} 1, & \text{if } \alpha \text{ is even;} \\ -1, & \text{if } \alpha \text{ is odd.} \end{cases}$$

e.g. 7. $n=2$. $\alpha = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \overset{\text{even}}{e}$ or $\overset{\text{odd}}{(12)}$

$\underline{\det A} = a_{11}a_{22} - a_{12}a_{21}$.

$n=3$

$$\alpha : \underbrace{(12),\ (13),\ (23)}_{\text{odd}},\ \underbrace{(123),\ (132),\ e}_{\text{even}}.$$

$$\det A = a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} + a_{11}a_{22}a_{33}$$
$$- a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22} - a_{11}a_{23}a_{32}$$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

How about $n=4$?