Week 10, 11

12,3 ; 12,8, 12,10

Subspaces

Definition

If W is a subset of a vector space V_F and W_F is a vector space over F (by using the same vector-addition), then W_F is called a vector subspace of V_F.

Example 1

Let $\mathbb{R}_1^n = \{ (0, a_2, a_3, \cdots, a_n) \mid a_i \in \mathbb{R} \text{ for } i = 2, \cdots, n \}$. Then $\mathbb{R}_1^n$ is a vector subspace of $\mathbb{R}^n$ (over $\mathbb{R}$).

Example 2

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then Ker(T) is a vector subspace of $\mathbb{R}^n$.

(*) If T is not a linear transformation, then Ker(T) may not be a vector subspace of $\mathbb{R}^n$.

e.g. Let $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x - 3$. Then $\text{Ker}(T) = \{3\}$ which is not a subspace of $\mathbb{R}$.

Theorem 1

W_F is a vector subspace of V_F if and only if $\forall \vec{u}, \vec{v} \in W$ and $\lambda \in F$, $\vec{u} + \lambda \vec{v} \in W_F$.

(Note) The proof provides a great way to check "subspace".

Proof. (⇒) Since W_F is a vector space itself, $\vec{u}+\lambda\vec{v}\in W_F$ for every $\vec{u}, \vec{v}\in W_F$ and $\lambda\in F$.

(⇐) There are quite a few conditions to check. The most important part is to show that $(W_F, +)$ is an abelian group, i.e.

(1) binary operation (let $\lambda = 1$).

(2) associative law (from V_F).

(3) identity $\vec{0}$ (from $\vec{v}, \vec{v}$, and $\lambda = -1$).

(4) inverse (from $\vec{u}+(-1)\vec{v}$ and $\vec{u}=\vec{v}$).

(5) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (from V_F).

The others are easy to see. ▨

Theorem 2 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\mathrm{Ker}(T)$ is a subspace of $\mathbb{R}^n$ denoted by $\mathrm{Ker}(T) \leq \mathbb{R}^n$.

Proof. Let $\vec{x}, \vec{y}\in \mathrm{Ker}(T)$ and $\lambda \in \mathbb{R}$. By Theorem 1, it suffices to show that $\vec{x}+\lambda\vec{y} \in \mathrm{Ker}(T)$. Now, consider $T(\vec{x}+\lambda\vec{y})$. Since T is a l. transf., $T(\vec{x}+\lambda\vec{y}) = T(\vec{x}) + T(\lambda\vec{y})$ $= T(\vec{x}) + \lambda T(\vec{y}) = \vec{0}+\lambda\vec{0} = \vec{0}$. This concludes the proof. ▨

Theorem 3 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a l. transf. Then $T(\mathbb{R}^n) = \{T(\vec{x}) \mid \vec{x}\in\mathbb{R}^n\} \leq \mathbb{R}^m$.

Proof. Since $T(\vec{x})+\lambda T(\vec{y}) = T(\vec{x}+\lambda\vec{y}) \in T(\mathbb{R}^n)$, the proof follows by Theorem 1.

Definition

Let T be a l. transf. from $\mathbb{R}^n$ into $\mathbb{R}^m$. Then $T(\mathbb{R}^n)$ is called the range space (subspace) of T from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Theorem 4

Let $\dim(Ker(T)) = n_0$. ~~and~~ Then $\dim(T(\mathbb{R}^n)) = n - n_0$.

Proof.

Let $B_1 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n_0}\}$ be a basis of $Ker(T)$. Then, we may add $n - n_0$ vectors $\vec{v}_{n_0+1}, \dots, \vec{v}_n$ to B_1 such that $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n_0}, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$. (?)

Now, we claim that $B' = \{T(\vec{v}_{n_0+1}), \dots, T(\vec{v}_n)\}$ is a basis of $T(\mathbb{R}^n)$.

Since T is a l. transf., B' is a l. independent set in $\mathbb{R}^m$.

($\lambda_1 T(\vec{v}_{n_0+1}) + \dots + \lambda_{n-n_0} T(\vec{v}_n) = \vec{0}$

$\Rightarrow T(\lambda_1 \vec{v}_{n_0+1} + \dots + \lambda_{n-n_0} \vec{v}_n) = \vec{0}$

$\Rightarrow \vec{u} = \lambda_1 \vec{v}_{n_0+1} + \dots + \lambda_{n-n_0} \vec{v}_n \in Ker(T)$. ($\vec{u}$ is a l. combination of $\vec{v}_1, \dots, \vec{v}_{n_0}$.)

Now, if some λ_i's are not equal to zero, then B is not an independent set. →← Hence ~~λ_{n_0+1}~~ $\lambda_1 = \dots = \lambda_{n-n_0} = 0$.)

Next $\{T(\vec{v}_{n_0+1}), \dots, T(\vec{v}_n)\}$ spans $T(\mathbb{R}^n)$.

$\forall\ \vec{y} \in T(\mathbb{R}^n)$, $\vec{y} = T(\vec{x}) = T(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n)$

$= \overset{\vec{0}}{\boxed{\alpha_1 T(\vec{v}_1) + \dots + \alpha_{n_0} T(\vec{v}_{n_0})}} + \alpha_{n_0+1} T(\vec{v}_{n_0+1}) + \dots + \alpha_n T(\vec{v}_n)$.

The rank of a matrix

对於任意给定的矩陣 $A = [a_{i,j}]_{m\times n}$ 我们可以讨论以下的三件事:

(1) 由列所生成的列空间，它的维数最多为 m，我们用 dim R(A) 来代表。(Row Space of A)

(2) 由行所生成的行空间，它的维数最多为 n，我们用 dim C(A) 来代表。(Column Space of A)

(3) 由 A 所对應的线性变换 T 的 Ker(T) 的维度，用 dim N(A) 来表示。(Null Space of A)

Theorem 4 Let A be an $m\times n$ matrix and suppose that A has been reduced to an echelon (reduced) matrix E.

(a) The nonzero rows of E forms a basis for $R(A)$.

(b) The columns of A corresponding to the pivot columns of E form a basis for the column space of A.

(c) $\text{rank}(A) = \dim R(A) = \dim C(A)$.

(d) nullity of $A = \dim N(A) = n - \text{rank}(A)$.

Proof.

(a) Since E is obtained from A by using row operations, $R(E) = R(A)$. By the fact that $R(E)$ is spanned by the set of nonzero rows, we have the proof.

(b) Observe the the pivot columns of E are l. independent and the other column can be obtained by a linear combination of these columns. Hence the corresponding columns of A form a basis of $C(A)$.

(c) $\text{rank}(A) = \dim R(A) = \dim C(A) =$ the # of pivot 1's.

(d) nullity of A = the # of free variables in solving the system of equations $A\vec{x} = \vec{0}$ which is also equal to $n -$ (the # of pivot 1's) $= n - \text{rank}(A)$. Hence

$$\text{nullity } A = n - \text{rank}(A).$$ ■

Corollary $\text{rank}(A) = \text{rank}(A^T)$.

Corollary $\text{rank}(A) \le \min\{m, n\}$ if A is an $m \times n$ matri