向量空间的基底 (Basis) 与维度 (Dimension)

Theorem If a vector space V has two bases S_1 and S_2 such that $|S_1| = m$ and $|S_2| = n$, then $m = n$.

Proof. Suppose not, let $m \neq n$ and $m < n$. Let $S_1 = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$ and $S_2 = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

Consider

$$A = [\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m].$$

By assumption that $\mathrm{Span}(S_1) = V$, $\vec{v}_i \in \mathrm{Span}(S_1)$ for $i = 1, 2, \cdots, n$. This implies that

$$A\vec{x} = \vec{v}_j, \ (j = 1, 2, \cdots, n) \text{ is consistent.}$$

Let $A\vec{b}_j = \vec{v}_j$ and $B = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n]$.

Hence $AB = C = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]$.

Since B is an $m \times n$ matrix, $m < n$, $B\vec{x} = \vec{0}$ has a nontrivial solution, i.e., not all coordinates are zero. Let $B\vec{u} = \vec{0}$ where $\vec{u} \neq \vec{0}$.

Now,
$$\begin{array}{c} AB\vec{u} = C\vec{u} = \vec{0}. \\ \| \\ A(B\vec{u}) = A\vec{0} = \vec{0} \end{array}$$

This implies that the columns of C are linearly dependent. $\rightarrow\leftarrow$.

Therefore $m = n$. ∎

<u>Note</u>

If S is a basis of V, then every vector of V has a unique representation (as a linear combination of the vectors in S).

Proof. Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

Let $\vec{v} \in V$ and

$$\vec{v} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_n \vec{v_n}$$
$$= \beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \cdots + \beta_n \vec{v_n}$$

This implies that

$$(\alpha_1 - \beta_1)\vec{v_1} + (\alpha_2 - \beta_2)\vec{v_2} + \cdots + (\alpha_n - \beta_n)\vec{v_n} = \vec{0}.$$

Since S is an independent set,

$$\alpha_i = \beta_i \text{ for } i = 1, 2, \cdots, n. \quad \blacksquare$$

Definition (Finite) (Dimension a vector space)

We say that a vector space V_F has dimension n (or that V_F is n-dimensional) if V has a basis consisting of n vectors. The dimension of V is denoted by $\dim V_F$.

(*) Suppose $\dim V_F = n$. Then no set of more than n vectors in V_F can be linearly independent and V_F can not spanned by fewer than n vectors.

(**) $\dim V_F$ is the maximum number of linearly independent vectors in V_F and also the minimum number of vectors needed to span V.

Theorem Suppose $\dim V_F = n$ and $S = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$

(1) If S is l. independent, then S is a basis of V_F.

(2) If $\mathrm{Span}(S) = V_F$, then S is a basis of V_F.

Proof. (1)

It suffices to prove that $\mathrm{Span}(S) = V_F$.

Let $\vec{v} \in V_F$ (任選).

Since $S \cup \{\vec{v}\}$ is linearly dependent,

$\exists\ \lambda_1, \lambda_2, \cdots, \lambda_{n+1}$, s.t. $\sum_{i=1}^{n+1} |\lambda_i| \neq 0$ and

$$\sum_{i=1}^{n} \lambda_i \vec{v}_i + \lambda_{n+1} \vec{v} = \vec{0}.$$

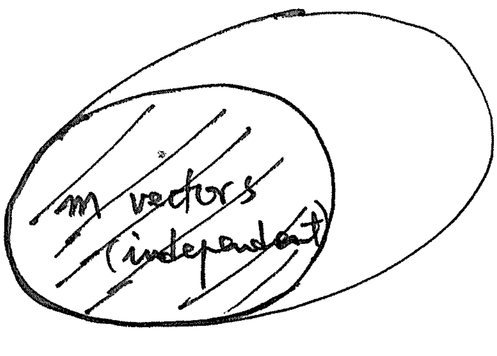
Now, $\lambda_{n+1} \neq 0$. For otherwise,

S is a linearly dependent set.

This implies that $\vec{v} = \sum_{i=1}^{n} \left(-\frac{\lambda_i}{\lambda_{n+1}}\right) \vec{v}_i \in \text{Span}(S)$. ∎

(***) <u>Ideas</u>

如果 V_F 的维度 n，则任意 $(m<n)$ 個独立向量都可以拿来扩充成一个基底。



$(1,0,0),\ (1,1,0) \in \mathbb{R}^3$

$S = \{(1,0,0),(1,1,0)\}$ is a l. independent set in $\mathbb{R}^3$.

Find a vector $\vec{v}$ which is not in Span(S).

$\Rightarrow$ $S \cup \{\vec{v}\}$ is a l. independent set in $\mathbb{R}^3$.

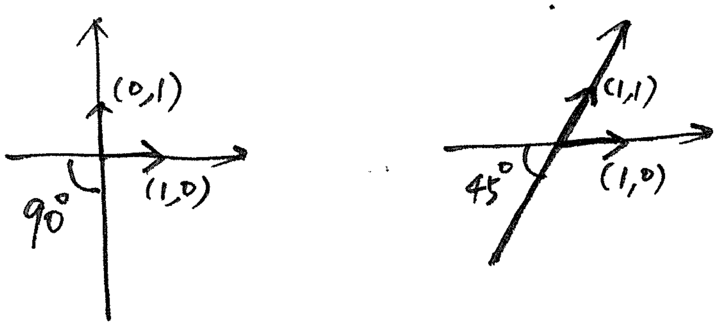
Since $|S \cup \{\vec{v}\}| = 3$ and $\dim(\mathbb{R}^3) = 3$, $S \cup \{\vec{v}\}$ is a basis of $\mathbb{R}^3$.

Conclusion

一個(有限維)向量空間可以有很多不同的基底，但是這些基底的向量數都是固定的。

Problem

不同基底所產生的現象如何？



線性轉換 (Linear Transformations)

(註) 在課本中 l. transf. 為了方便於解釋，只考慮從 $\mathbb{R}^n$ 對應到 $\mathbb{R}^m$ 的映射。

(*) 由於 $\mathbb{R}^n$, $\mathbb{R}^m$ 都可以是佈於 $\mathbb{R}$ 的向量空間，以下的定義都是考量一般的向量空間。

Definition (L. Transf.)

Let V_F and W_F be two vector spaces. A function T from V_F into W_F is called a linear transf. if for any two vectors $\vec{x}$ and $\vec{y}$, and scalar $\lambda \in F$,

$$T(\vec{x}+\lambda\vec{y}) = T(\vec{x}) + \lambda T(\vec{y}).$$

(Note) "$T(\vec{x}+\lambda\vec{y}) = T(\vec{x}) + \lambda T(\vec{y})$" can also be "$T(\mu\vec{x}+\lambda\vec{y}) = \mu T(\vec{x}) + \lambda T(\vec{y})$" or "(1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$".

"Linear" 的概念可以推廣至线性组合的对應。

令 $\mathbb{B}$ 为 V_F 的一个基底(basis), $\mathbb{B}=\{\vec{b}_1,\vec{b}_2,\cdots,\vec{b}_n\}$.

则 $\forall\ \vec{v}\in V_F$, $\vec{v}=\sum_{i=1}^{n}\lambda_i\vec{b}_i$, $\lambda_i\in F$.

$T: V_F \longrightarrow W_F$ (l. transf.)

$$\Rightarrow\ T(\vec{v})=T\left(\sum_{i=1}^{n}\lambda_i\vec{b}_i\right)=\sum_{i=1}^{n}\lambda_i T(\vec{b}_i).$$

(***) 如果 T 为一线性轉換,则

$T(\vec{b}_i)$, $i=1,2,\cdots,n$, 可以决定 $T(\vec{v})$, $\forall\ \vec{v}$.

亦即 T 可以经由基底的对應来决定之。

Example 1

T 为 $\mathbb{R}^2$ 映至 $\mathbb{R}^2$ 的线性轉換,且 $T(1,0)=(5,3)$, $T(0,1)=(2,1)$, 则 $T(-1,2)=T\big((-1)(1,0)+2\cdot(0,1)\big)$

$$=-T(1,0)+2T(0,1)$$

$$=(-5,-3)+(4,2)$$

$$=(-1,-1).$$

∎

Example 2

Let A be an $m \times n$ matrix. Then $T(\vec{x}) = A\vec{x}$ $\forall$ $\vec{x} \in \mathbb{R}^n$ is a linear transf. from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Example 3

The following transformation is not linear.

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ x_1 - x_2 + \underline{\underline{3}}).$$

(*) 一般函数的对应如 $f: \mathbb{R} \to \mathbb{R}$ 也可以看成由 v. space 对到 v. space 的 transf. 如此一来判断是否 linear 就很直观。

例如： $f(x) = 3x$. (Yes!)

$$f(x + \lambda y) = 3(x + \lambda y) = 3x + \lambda 3y$$
$$= 3f(x) + \lambda f(y).$$

$f(x) = x^2$ (No!)

$$f(x + \lambda y) = (x + \lambda y)^2 = x^2 + 2\lambda xy + \lambda^2 y^2$$
$$\neq f(x) + \lambda f(y)$$ (For some x, y.)

$f(x) = 3x + 2$. (No?)

Example 4

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by rotating the vector (a,b) an angle θ counterclockwisely, see Fig. 1

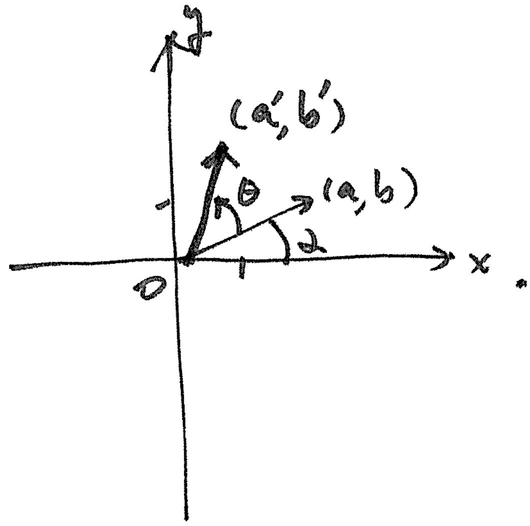


Figure 1

Then T is a l. transf.

By direct calculation,

$$T(a,b) = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= (-a\cos\theta + b\sin\theta,\ a\sin\theta + b\cos\theta) \quad (?)$$

Hint: Let $|(a,b)| = r$. Then $(a,b) = (r\cos\alpha, r\sin\alpha)$.

Since $T =_{\text{def}} \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, T is a l. transf. ∎

Example 5

Let D be the set of all differentiable functions with domain $\mathbb{R}$ and codomain $\mathbb{R}$. Then the function D_x defined by $D_x(f) = f'$ is a l. transf. from D into $\mathbb{R}^{\mathbb{R}}$ (functions with domain $\mathbb{R}$ and codomain $\mathbb{R}$).

Example 6

Let C be the set of continuous functions from $[a,b]$ into $\mathbb{R}$. Then $T(f) = \int_a^b f(x)\,dx$ is a l. transf. from C into $\mathbb{R}$.

(註) Both D and C are vector spaces over $\mathbb{R}$.

Example 7

Let T be a l. transf. from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $T(1,0) = (1,0)$ and $T(0,1) = (1,1)$. Then T can be obtained by the following way.

(*) $T \approx \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases} \Rightarrow \boxed{\begin{matrix} a=1,\ b=1 \\ c=0,\ d=1 \end{matrix}}$