Week 5 10,13 and 10,15

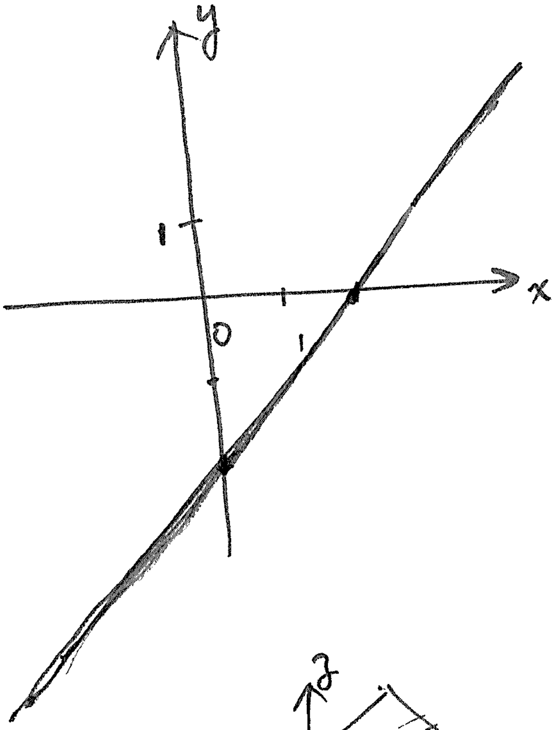
從幾何的觀點看聯立方程組(線性規劃)

一個 n 元線性方程式

$$a_1x_1+a_2x_2+\cdots+a_nx_n=c,\quad a_i\in\mathbb{R},\ c\in\mathbb{R}$$

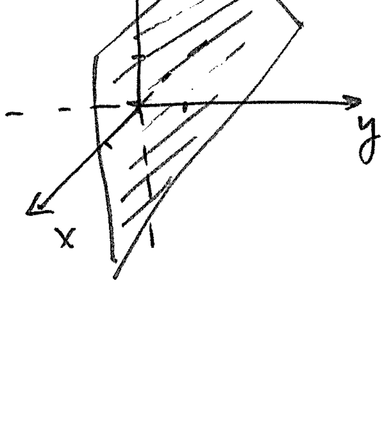
可以看成是 $\mathbb{R}^n$ 中的一個 **hyperplane** (超平面),

(自由度為 $n-1$).



$x-y=2$

(自由度為 1).



$ax+by+cz=d$

(自由度為 2)

联立方程组

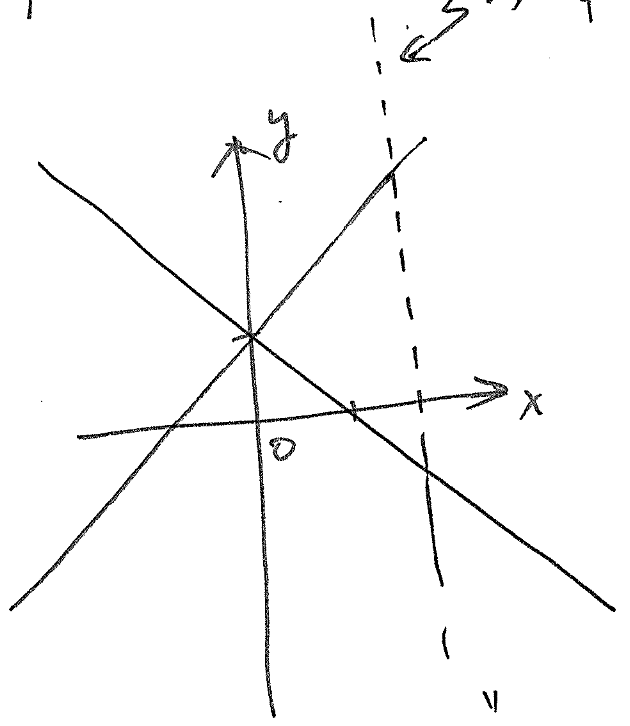
$$\begin{cases} ① \\ ② \\ ③ \end{cases}$$

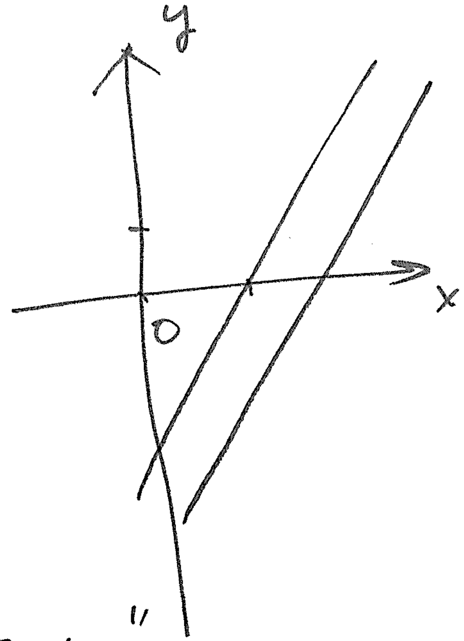
代表解要同时是①,②,③的解。

令 S_1, S_2, S_3 分别代表解集合,

则 $S = S_1 \cap S_2 \cap S_3$ 为联立方程组的解集合。

如果多了这条线,则<u>无解</u>。





" or <u>重合</u> "

例：在 3-dim. $\mathbb{R}^3$，

联立方程组是探讨平面之间的"交集"問题。

例：在 n-dim. $\mathbb{R}^n$

探讨 (n-1)-dim. hyperplanes 之间的"交集"問题。

交集的基本概念

$$A_1 \cap A_2 \subseteq A_1 , \quad A_1 \cap A_2 \subseteq A_2$$

$$I_1 \subseteq I_2$$

$$\bigcap_{i \in I_1} A_i \supseteq \bigcap_{j \in I_2} A_j$$

$$\Rightarrow \bigcap_{i \in \phi} A_i \supseteq S , \quad (\forall S).$$

(*) Index 的集越大，解集合越小。

回推到聯立方程組

方程的個數少於(或等於)未知數的個數就比較容易有解，

但是"不一定會有解"。

例：

$$\begin{cases} 3x+5y+2z=10 \\ 6x+10y+4z=5 \end{cases}$$

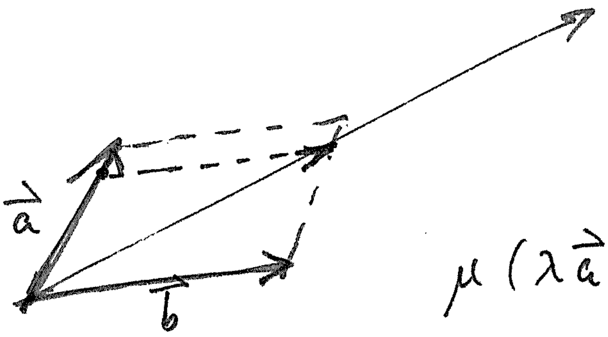
就無解！(why?)

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 10 \\ 6 & 10 & 4 & 5 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 10 \\ 0 & 0 & 0 & -15 \end{array}\right] \longleftarrow \text{無解的形式}$$

向量的线性组合



$\mu(\lambda\vec{a}+\vec{b})$

① 一个向量决定一直线（通过 (0,0,⋯)）.
原点

② 二个向量"可以"决定一平面（If independent）.

⋮

何時 $A\vec{x}=\vec{b}$ 一定有解？

(*) 令 $A=[\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]_{m\times n}$.

$\vec{b}\in \mathrm{Span}(\{\vec{a}_1,\vec{a}_2,\cdots,\vec{a}_n\})$

$\Rightarrow A\vec{x}=\vec{b}$ 有解。

(*) $\vec{a}_1,\vec{a}_2,\cdots,\vec{a}_n\in\mathbb{R}^m$.

$\Rightarrow \mathrm{Span}(\{\vec{a}_1,\vec{a}_2,\cdots,\vec{a}_n\})\subseteq\mathbb{R}^m$.

(**) Let $S=\mathrm{Span}(\{\vec{a}_1,\vec{a}_2,\cdots,\vec{a}_n\})$.

$\forall\ \vec{b}\in\mathbb{R}^m\setminus S$, $A\vec{x}=\vec{b}$ 無解！

例

$A=\begin{bmatrix}3&5&2\\6&10&4\end{bmatrix}$, $\mathrm{Span}\left(\left\{\begin{pmatrix}3\\6\end{pmatrix}^T,\begin{pmatrix}5\\10\end{pmatrix}^T,\begin{pmatrix}2\\4\end{pmatrix}^T\right\}\right)$

$(3,6),(5,10),(2,4)$

$=\lambda\cdot(1,2),\ \lambda\in\mathbb{R}$.

$(10,5)\notin\{\lambda\cdot(1,2)\mid\lambda\in\mathbb{R}\}\Rightarrow A\vec{x}=\begin{pmatrix}10\\5\end{pmatrix}$ 無解。

何時一定有解？

$$\text{Span}(\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}) = \mathbb{R}^m.$$

$\Longleftrightarrow$ $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ 中有 m 個向量線性独立。

2-dim.

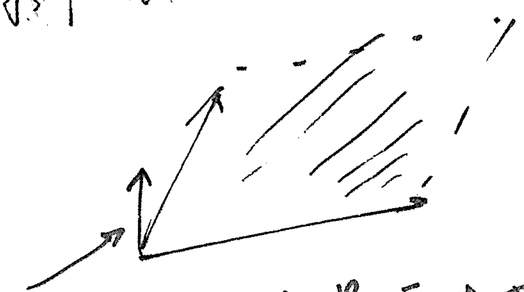
有兩个向量線性独立：

不平行且不重合的兩個向量，

的夾角 $0^\circ < \theta < 180^\circ$.

3-dim.

有三个向量線性独立：



不在由二个向量所生成的平面上。

Proposition

Let $n > m$. Prove that any n vectors in $\mathbb{R}^m$ form a linearly dependent set.

Review

(Fact 1) If a set of vectors contains $\vec{0}$, then the set of vectors forms a l. dependent set.

(Fact 2) Let $A \subseteq B$ be a set of vectors. Then A is linearly independent provided B is l.ind..

(Fact 3) Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a set of l. independent vectors and $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ are in $\mathbb{R}\setminus\{0\}$. Then $\{\vec{v}_1 + \lambda_1\vec{v}_n, \vec{v}_2 + \lambda_2\vec{v}_n, \cdots, \vec{v}_{n-1} + \lambda_{n-1}\vec{v}_n\}$ is a l. independent set.

Proof. Suppose not. Then, there exist $\mu_1, \mu_2, \cdots, \mu_{n-1}$ (not all zeros) such that

$$\sum_{i=1}^{n-1} \mu_i (\vec{v}_i + \lambda_i \vec{v}_n) = \vec{0}$$

$$\sum_{i=1}^{n-1} \mu_i \vec{v}_i + \left(\sum_{i=1}^{n-1} \mu_i \lambda_i\right) \vec{v}_n = \vec{0}$$

$\Rightarrow \{\vec{v}_1, \cdots, \vec{v}_n\}$ is l. dependent. $\rightarrow\leftarrow$ ■

(Fact 4)

Let $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$ be m vectors in R^m such that the mth coordinate of those vectors is 0. Then $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\}$ is l. dependent.

Proof. By induction on m.

Clearly, it is true for $m = 1, 2$.

Assume that the assertion is true for k and let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{k+1}\}$ be $k+1$ vectors in R^{k+1} with 0 in the last coordinate such that it is a l. independent set.

Therefore $\vec{u}_{k+1} \neq \vec{0}$. Let the k^{th} coordinate of $\vec{u}_{k+1}$ is not equal to 0. Now, we can find $\mu_1, \mu_2, \cdots, \mu_k$ such that the k^{th} coordinate of $\mu_i \vec{u}_{k+1} + \vec{u}_i$ is zero.

By (Fact 3) $\{\vec{u}_1 + \mu_1 \vec{u}_{k+1}, \cdots, \vec{\mu}_k + \mu_k \vec{u}_{k+1}\}$ is l. indep.

But this is not possible —