Week 4 , 10,6 ; 10,8

$\boxed{\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \text{ in } R^m}$

如何决定一组向量是相依或独立？

(Fact 1) 有一 $\vec{0}$ 在裏面 $\Rightarrow$ 相依。

(Fact 2) 有一向量為另一向量的倍数，$\vec{v}_i = \lambda \vec{v}_j$，$\lambda \neq 0$

$\Rightarrow$ 相依。

(Fact 3) 一向量為另外 n 个向量的線性組合，

$\Rightarrow$ 相依。

(Fact 4) $n > m \Rightarrow$ 相依 (?) $\left[\begin{matrix} \text{Rank}(A) \leq m \\ < n \end{matrix} \right]$

(見下页)

(Fact 5) 看 $A\vec{x} = \vec{0}$！

有唯一解 $\vec{0}$，则独立，否则相依。

(Fact 6) 直接看 $\sum_{i=1}^{n} \lambda_i \vec{v}_i = \vec{0}$ 是否有

$(\lambda_1, \lambda_2, \ldots, \lambda_n)$ 满足此方程式。

2. Rank

下列 6 个叙述等价：

1. $\text{Rank}(A_{m\times n}) < n$.

2. The reduced row echelon form of A, $\tilde{A}$, has fewer than n nonzero rows.

3. The matrix $\tilde{A}$ has fewer than n pivot positions.

4. At least one column of $\tilde{A}$ has no pivot position.

5. There is at least one free variable in $A\vec{x} = \vec{0}$.

6. The system $A\vec{x} = \vec{0}$ has some nontrivial solutions.

7. The columns (vectors) are linearly dependent.

$$A\vec{x} = \vec{b}$$

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\uparrow$ (m-dim. vector)

Each vector in the span of a linearly independent set has a unique representation as a linear combination of elements of that set.

$A\vec{x} = \vec{b}$ 有解

$\Leftrightarrow \vec{b} \in \mathrm{Span}(\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\})$

何時 $\mathrm{Span}(\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}) = \mathbb{R}^m$?

① $n < m$，不可能！

② $n \geq m$；不一定，成立的條件為

$\mathrm{Rank}(A) = m$!

Definition (Rank of matrix)

The rank of a matrix is the number of nonzeros in its "reduced row echelon form". We use Rank(A) to denote this number.

(*)

$\text{Rank}(A) \le \min(m, n)$.

(*)

$\text{Rank}(A) < n \iff A\vec{x} = \vec{0}$ has some nontrivial solutions(or

$\text{Ker}(A) = \text{Null}(A) \supsetneq \{\vec{0}\}$.