Week 3 9.29-10.1

提醒：解联立方程组是这一章的主题，莫忘了找一些实例练习一下！

实际上去解联立方程组要用"列运算"

① Scale operation

把一个 scalar 乘在一列中的每一个数。

② Interchanging operation

换列。

③ Replacement operation

$$(i)\cdot\lambda + (j) \longrightarrow (j)$$

第 i 列　　第 j 列

row equivalent

$A \sim B$

if $A \to \to \cdots \to B$

① ② or ③

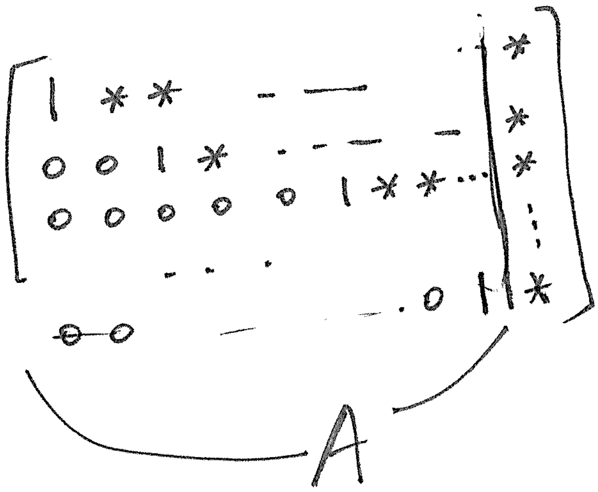
<u>Theorem</u> If two augmented matrices are <u>row equivalent</u> to each other, then the solutions of the two systems are identical.

After applying row-reduced operations

(reduced row echelon form).

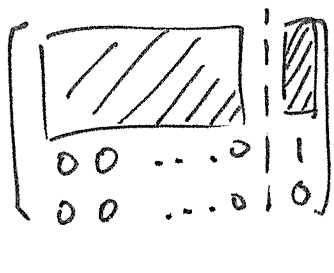
<u>Theorem</u> (有解)

Let A be an $m \times n$ matrix. The system of equations $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$ if and only if each row of the coefficient matrix A ~~has~~ ~~be~~ pivot position after applying row operations.



<u>Theorem</u> (無解)

A system of linear equations is inconsistent if and only if its augmented matrix has a pivot position in the last column.



從另一个角度来看有没有解。

$$\underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{m\times n}\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

令 A 的行(向量)为 $\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n$。

$A\vec{x} = \vec{b}$ has a solution $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ if and only if $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$.

Let $S = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$. Then $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b} \in \mathrm{Span}(S)$.

$A\vec{x} = \vec{b}$ is a homogeneous linear system if $\vec{b} = \vec{0}$.

<u>Definition</u> A set of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is said to be linearly dependent if $\exists\ \lambda_1, \lambda_2, \cdots, \lambda_n \in R$ such that $\sum_{i=1} \lambda_i \vec{v}_i = \vec{0}$ where $\sum_{i=1}^{n} \lambda_i^2 \neq 0$.

<u>Facts</u> ① If one of $\vec{v}_1, \cdots, \vec{v}_n$ is a zero vector, then $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a set of linear dependent vectors.

<u>Definition</u> If S is not l. dependent, then S is l. independent.

② If $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is l. dependent, then some vector in S is a l. combination of the other vectors in S.

Both ① and ② can be proved!

How to prove S is a l. <u>independent set</u> ?

(*) Let $\sum_{i=1}^{n} \lambda_i \vec{v}_i = \vec{0}$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

(*) Theorem $A\vec{x} = \vec{0}$ has unique solution $\vec{0}$ if and only if the columns (vectors) are l. independent.

(**) 如何 check "columns are dependent or not"?

Sol. 可以用 reduced row-echelon form!

Theorem The columns of a matrix form a l. independent set if and only if there is a pivot position in each column of a row echelon form of the matrix.

Notice

Rows can also be considered as vectors!

(要說明是列或是行。)

<u>Definition</u> (Kernel) of A.

The set (solution) of $\vec{x}$ s.t. $A\vec{x}=\vec{0}$ is called the kernel of A, denoted by $\mathrm{Ker}(A)=\{\vec{x} \mid A\vec{x}=\vec{0}\}$.

(註) 因为 $\vec{x}=\vec{0}$ 为解，所以 $\mathrm{Ker}(A)\neq\phi$.

$$A=\begin{bmatrix}1&3&5\\2&1&4\end{bmatrix}$$

$$\begin{bmatrix}1&3&5\\2&1&4\end{bmatrix}\begin{bmatrix}7\\6\\-5\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}7\\6\\-5\end{bmatrix} \text{ or } (7,6,-5)\in \mathrm{Ker}(A).$$

(註) 把 A 想成是一个函数，则所有函数值为 $\vec{0}$ 的向量组合成 $\mathrm{Ker}(A)$.

Kernel 具有很好的性质!

$\vec{x}, \vec{y} \in \mathrm{Ker}(A) \Rightarrow \vec{x}+\vec{y} \in \mathrm{Ker}(A)$

$a \in \mathbb{R}, \vec{x} \in \mathrm{Ker}(A) \Rightarrow a\vec{x} \in \mathrm{Ker}(A)$

$\vec{x}+\lambda\vec{y} \in \mathrm{Ker}(A)$

向量子空间 (Vector subspace)!

Fact

If $A\vec{x}=\vec{b}$ and $\vec{y} \in \mathrm{Ker}(A)$,

then $A(\vec{x}+\vec{y})=b$.

($\vec{x}$为$A\vec{x}=\vec{b}$的解, $\Rightarrow$ $\vec{x}+\vec{y}$也是解!).

Fact

$A\vec{x}=A\vec{y} \Rightarrow \vec{x}-\vec{y} \in \mathrm{Ker}(A)$

($\vec{x},\vec{y}$都是$A\vec{x}=\vec{b}$的解 $\Rightarrow$ $\vec{x}-\vec{y} \in \mathrm{Ker}(A)$.)

More important fact

If $\vec{u}$ is a solution of $A\vec{x}=\vec{b}$, i.e. $A\vec{u}=\vec{b}$, then $\forall\, \vec{y}$ satisfying $A\vec{y}=\vec{b}$, $\vec{y}=\vec{u}+\vec{z}$, $\vec{z}\in \mathrm{Ker}(A)$.

(註) 先求 $A\vec{x}=\vec{0}$ 的解集合 $\mathrm{Ker}(A)$.

再求 $A\vec{x}=\vec{b}$ 的特殊解 $\vec{u}$.

$\Rightarrow$ $A\vec{x}=\vec{b}$ 的一般解(集合)為

$$\vec{u}+\mathrm{Ker}(A)$$

$$=\left\{\vec{u}+\vec{z} \;\middle|\; \vec{z}\in \mathrm{Ker}(A)\right\}.$$

(註) 對於 $A\vec{x}=\vec{b}$ 的所有解也都可以寫成 $\vec{u}+\mathrm{Ker}(A)$ 的形式，其中 $A\vec{u}=\vec{b}$.

Proof. 令 $\vec{y}$ 為 $A\vec{x}=\vec{b}$ 的另一解，則 $A(\vec{y}-\vec{u})=\vec{b}-\vec{b}=\vec{0}$，所以 $\vec{y}-\vec{u}\in \mathrm{Ker}(A)$.