判断一个联立方程组是否有解，除了具正地用消去法算之看之外，也可以利用向量之间的关系来判断。

先看一个例子

$$\begin{cases} 2x+3y=5 \\ 4x-y=3 \end{cases}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$x\begin{bmatrix} 2 \\ 4 \end{bmatrix}+y\begin{bmatrix} 3 \\ -1 \end{bmatrix}=\begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \text{———— (1)}$$

有解：$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ 可以写成 $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ 与 $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ 的线性组合。

（充要条件）

找到 $x, y \in \mathbb{R}$ 满足 (1)。

但是 $\begin{bmatrix}5\\3\end{bmatrix}$ 無法寫成 $\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}$ 的線性組合, why?

<u>Definition</u> (Linear Combination)

A vector $\vec{w}$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ if it can be expressed in the form

$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$

where α_i is a <u>scalar</u> ($\alpha_i \in \mathbb{R}$ for example), $i = 1, 2, \cdots, n$.

<u>Question</u>

Is $(-1, 3, 7)$ a linear combination of $(4, 2, 7)$ and $(3, 1, 4)$?

Solve the system of linear equations

$$\begin{cases} 4x + 3y = -1 \\ 2x + y = 3 \\ 7x + 4y = 7 \end{cases} \Rightarrow x = 5, \ y = -7.$$

(不一定每次都如此幸運!)

(Fact)

(1) Every vector in $\mathbb{R}^2$ is a linear combination of $\underset{\vec{e}_2}{\underset{\|}{(0,1)}}$ and $\underset{\vec{e}_1}{\underset{\|}{(1,0)}}$.

(2) Every vector in $\mathbb{R}^n$ is a l.c. of

$$\vec{e}_1 = (1,0,\cdots,0),\ \vec{e}_2 = (0,1,0,\cdots,0),\ \cdots,\ \vec{e}_n = (0,0,\cdots,0,1)$$

where $\vec{e}_i$ is an n-dim. vector.

(3) Every vector in $\mathbb{R}^2$ is a l.c. of $(5,2)$ and $(7,3)$. (?)

不相信可以解联立方程组。

(4) $(1,2)$ and $(2,4)$ 不可能把 $\mathbb{R}^2$ 中的向量都表示出来，例如 $(1,3)$ 就无法写成 $(1,2)$ 及 $(2,4)$ 的 l.c. (?)

Definition (Span)

Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$. The set of all l.c., denoted by $\mathrm{Span}(S)$, of vectors in S is called the span of S.

Example Let $S = \{(1,2), (3,5)\}$.

Then $\mathrm{Span}(S) = \mathbb{R}^2$. (Why?)

Theorem (联立方程组的第一定理)

A system of linear equations $A\vec{x} = \vec{b}$ is consistent if and only if the vector $\vec{b}$ is in
(有解)
the span of the set of columns of A.

(註) A 的 columns 可以看成向量!

Proof.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(Span Form)

$\Longleftrightarrow$ $A\vec{x} = \vec{b}$ is consistent.