Chapter 3

More General Designs

To combinatorialist, a design is usually a 2-design which is a balanced incomplete block design or a pairwise balanced design. However, 2-designs do not necessarily exist in all cases where a researcher might wish to use one to design an experiment. As a consequence, we need to consider structures much more general than the combinatorialist's designs, and decide which one is more proper in a given situation. This leads to the theory of optimal design studied by statisticians. In this chapter, we shall not go into the details of optimality of a design. Instead, we introduce several types of designs which play key roles in the study of combinatorial designs.

3.1 Graph Designs

Definition 3.1.1. If H can be decomposed into isomorphic copies of G, then we say "a G-design of H" exists, denoted by $G \mid H$.

Definition 3.1.2. If $G \mid K_n$, then we say " a G-design of order n" exists.

Lemma 3.1.3. *If $G \mid H$ and their degree sequences are $\{d_1, d_2, \cdots, d_k\}$ and $\{f_1, f_2, \cdots, f_n\}$ respectively, then (i) $|G| \leq |H|$, (ii) g.c.d.$\{d_1, d_2, \cdots, d_k\}$ divides f_i for each $i = 1, 2, \cdots, n$, and (iii) $\|G\| \mid \|H\|$.*

Theorem 3.1.4. *A C_4-design of order n exists if and only if $n \equiv 1 \pmod 8$.*

Proof.

(Method 1) By using $v \to v + 8$ construction.

First, it is not difficult to see that K_9 (defined on $\mathbb{Z}_9$) can be decomposed into C_4's by using difference method with the base cycle (0,1,5,2). Now, assume that K_v (defined on $\mathbb{Z}_v$) can be decomposed into C_4's and $\{y_i \mid i \in \mathbb{Z}_8\}$ is a set of 8 vertices which is disjoint with $\mathbb{Z}_v$. Now, let α be the collection of C_4's obtained from K_v (defined on $\mathbb{Z}_v$), β be the collection of C_4's obtained from K_9 (defined on $\{0\} \cup \{y_i \mid i \in \mathbb{Z}_8\}$ and γ be the collection of C_4's obtained from $K_{v,8}$ defined on $\mathbb{Z}_v \cup \{y_i \mid i \in \mathbb{Z}_8\}$. Then, we have $C_4 \mid K_{v+8}$.

(Method 2) Cyclic C_4-design

Let $v = 8k + 1$. Therefore the difference set is $\{1, 2, \cdots, 4k\}$. Now, for each $1 \leq j \leq k$, let $C^{(j)} = (0, 4j-3, 8j-3, 4j-2)$. Then, it is not difficult to check $\{C^{(j)} \mid j = 1, 2, \cdots, k\}$ form a collection of base cycles of K_{8k+1}. Hence, $C_4 \mid K_v$. ∎

We can of course extend our study to cycles of other length, but we will focus on 4-cycles. In what follows, we consider the existence of C_4's in a bipartite graph.

Definition 3.1.5. An *H-subgraph* of a graph is a subgraph that is isomorphic to a given graph H. A graph is *H-free* if it contains no H-subgraphs. An H-free subgraph G of a graph K is *H-saturated* if it has the additional property that for any edge $e \in E(K) \setminus E(G)$, $G \cup \{e\}$ is not H-free.

Here, we are mostly interested in C_4-free subgraphs of $K_{m,n}$. First, we need a design and a graph to tackle this problem.

Definition 3.1.6. A *(partial) t-(v, K, λ) design* is a pair (V, B) where V is a v-set and B is a collection of subsets, with sizes belonging to K, of V with the property that every t-set of V is in (at most) exactly λ subsets of B.

Definition 3.1.7. Given any partial t-(v, K, λ) design (V, B), the *variety-block graph* of the design is the bipartite graph $G_{V,B}$ with vertex partition V, B defined by joining $v \in V$ to $b \in B$ if and only if $v \in b$.

Clearly, if $G_{V,B}$ contains a $K_{t,\lambda+1}$-subgraph, with the part of size t in V and the part of size $\lambda + 1$ in B, then there will be a t-set which occurs in $\lambda + 1$ blocks. Conversely, given a $K_{t,\lambda+1}$-free bipartite graph $G_{V,B}$, with vertex partition V, B, we can define a partial t-(v, K, λ) design (V, B) by letting $v \in b$ if and only if $v \in V$ is joined to $b \in B$ in $G_{V,B}$.

Definition 3.1.8. A partial 2-$(v, K, 1)$ design (V, B') is said to be an *extension* of a partial 2-$(v, K, 1)$ design (V, B) if $B \neq B'$ and for each $b \in B$ there exists a $b' \in B'$ with $b \subseteq b'$. A partial 2-$(v, K, 1)$ design is said to be *non-extendable* if it has no extension.

The following lemmas are easy to check.

Lemma 3.1.9. *The variety-block graph $G_{V,B}$ of a partial 2-$(v, K, 1)$ design (V, B) is a C_4-saturated subgraph of $K_{|V|,|B|}$ if and only if (V, B) is non-extendable.*

Lemma 3.1.10. *A C_4-saturated subgraph of $K_{m,n}$ contains at least $m + n - 1$ edges.*

Lemma 3.1.11. *For all $m, n \geq 2$, there exists a C_4-saturated subgraph of $K_{m,n}$ with $m+n-1$ edges.*

Now, we have a good theorem to find a C_4-saturated subgraph of $K_{m,n}$ with maximum size.

Theorem 3.1.12. *Suppose (V, B) is a partial 2-$(v, K, 1)$ design with $B = \{b_1, b_2, \cdots, b_n\}$. If $m = v$, $K = \{k, k + 1\}$ for some positive integer k, and $\binom{v}{2} - \sum_{i=1}^{n} \binom{|b_i|}{2} < k$ then the variety-block graph of (V, B) is a C_4-saturated subgraph of $K_{m,n}$ having maximum size.*

Proof. Omitted. ∎

Corollary 3.1.13. *Let $G_{V,B}$ be the variety-block graph of a 2-$(m, \{k, k+1\}, 1)$ design (v, B) with $|B| = n$. Then $G_{V,B}$ is a C_4-saturated subgraph of $K_{m,n}$ with maximum size.*

Reference: *Discrete Math.* **259** (2002), 263–268.

3.2 Group Divisible Design (GDD)

Definition 3.2.1. (k-GDD)
A group-divisible design $GD[n, m; k; \lambda]$ is an ordered triple $(V, \mathcal{G}, \mathbb{B})$, where V is a set of mn elements, $\mathcal{G}$ is a partition of V into m groups of size n, and $\mathbb{B}$ is a collection of k-subsets of V called blocks such that:
(1) each pair of elements that occur in the same group, occur together in no blocks, and
(2) each pair of elements that occur in different groups, occur together in exactly λ blocks.

The class of $GD[n, m; k; \lambda]$ is called a k-GDD (for brevity).
(**Note 1**) If we replace the frequency λ by λ_1 and λ_2 in (1) and (2) respectively, then we have a $GD[n, m; k; \lambda_1, \lambda_2]$ which is known as a k-GDD with two associates.
(**Note 2**) The existence of a $GD[n, m; k; \lambda]$ is equivalent to the existence of $K_k \mid \lambda K_{m(n)}$, i.e., $\lambda K_{m(n)}$ can be decomposed into K_k's.
(**Note 3**) A $GD[n, m; k; \lambda]$ is a transversal design if $k = m$, denoted by $T[n; k; \lambda]$.

In this section, we shall focus on 3-GDD.
Before we prove the main result, we need several lemmas.

Lemma 3.2.2. *If an $STS(m)$ exists, then a $GDD(n, m; 3; 1)$ exists.*

Proof. It is well-known that the existence of a latin square of order n implies the existence of $K_3 \mid K_{3(n)}$. Therefore, we first consider the m groups as m vertices and define an $STS(m)$ $(V', \mathbb{B}')$ on these m vertices. Then, for each triple in $\mathbb{B}'$ we have a $K_{3(n)}$ in $K_{m(n)}$. By defining triples in $K_{3(n)}$ from a latin square of order n, we have a K_3-decomposition of $K_{m(n)}, \mathbb{B}$, such that each pair of elements from different groups occur together in exactly one triple of $\mathbb{B}$. ∎

Theorem 3.2.3. *$K_4 \mid K_v$ if and only if $v \equiv 1$ or $4 \pmod{12}$.*

Proof. The necessity is not difficult to see but the sufficiency of the statement does take some effort to get it done. We omit the detail here. ∎

Reference: J. C. Bermond and J. Schönhein, "G-decomposition of K_n, where G has four vertices or less," *Discrete Math.* **19** (1977), 113–120.

Theorem 3.2.4. *A $GD[n, m; 3; 1]$ exists if and only if $m \neq 2$ and $K_{m(n)}$ is 3-sufficient.*

Proof. (⇒) The following table depicts the relationship between m and n if $K_{m(n)}$ is 3-sufficient.

$m \backslash n$	0	1	2	3	4	5 (mod 6)
0	②	×	②	×	②	×
1	①	①	①	①	①	①
2	④	×	×	×	×	×
3	①	①	①	①	①	①
4	②	×	②	×	②	×
5	③	×	×	③	×	×

(mod 6) $m > 2$

× : The cases which are not possible.
ⓘ : Obtained by construction "ⓘ".

(⇐)

Case ①. It follows from Lemma 3.2.2, since a latin square of order n exists for each $n \in \mathbb{N}$.

Case ②. Let $n' = \frac{n}{2}$ and each group be partitioned into two sub-groups each of size n'. Since $K_{2m} - F$ can be decomposed into triangles, the 3-GDD can be obtained by "growing" each vertex into a set of n' vertices and define n'^2 triples by using a latin square of order n' for each triangle obtained from $K_3 \mid K_{2m} - F$. (Here the groups of $K_{m(n)}$ are corresponding the union of two sub-groups obtained from an edge of F.)

Case ③. By the idea of Lemma 3.2.2, it suffices to prove this case for $m = 5$. (?) Now, partition each group into 3 sub-groups and apply a $KTS(15)$ with the above idea. Here, we reserve a parallel class to keep the 5 groups with no edges in between.

Case ④. Since a $GD[6, m; 3; 1]^{(*)}$ exists for every $m \geq 3$, the proof follows by grouping k vertices together, i.e., a $GD[6k, m; 3; 1]$ exists. This completes the proof of this case and theorem. ∎

(∗) Reference: Lemma 6.5, page 356, *DM* **11** (1975), 255–369.

3.3 Transversal Design

A transversal design with s groups of size r and frequency λ denoted by $T[r, s; \lambda]$ is a $GD[r, s; s; \lambda]$, i.e., a group divisible design with block size $k = s$ (the number of groups). For convenience, we use $T(s, \lambda)$ to denote the set of r's such that a $T[r, s; \lambda]$ exists.

The followings are not difficult to verify.

Lemma 3.3.1. *If $s' \leq s$, then $T(s, \lambda) \subseteq T(s', \lambda)$.*

Lemma 3.3.2. *If $\lambda' \mid \lambda$, then $T(s, \lambda') \subseteq T(s, \lambda)$.*

Lemma 3.3.3. *If m, n are positive integers, then $T(s, \lambda) \cap T(s, \lambda') \subseteq T(s, m\lambda + n\lambda')$.*

Lemma 3.3.4. *If $r \in T(s, \lambda)$ and $r' \in T(s, \lambda')$, then $rr' \in T(s, \lambda\lambda')$.*

Proof. First, we have a transversal design $T[r', s; \lambda']$ by assumption. Then, for each block in $T[r', s; \lambda']$ we replace it with a $T[r, s; \lambda]$ by blowing up each point to r points. ∎

Definition 3.3.5. A transversal design $T[r, s; \lambda]$ is *resolvable* if the collection of blocks can be partitioned into λr parallel classes, denoted by $RT[r, s; \lambda]$ and the set of all r's such that an $RT[r, s; \lambda]$ exists is denoted by $RT[s, \lambda]$.

Lemma 3.3.6. $RT(s, 1) = T(s + 1, 1)$.

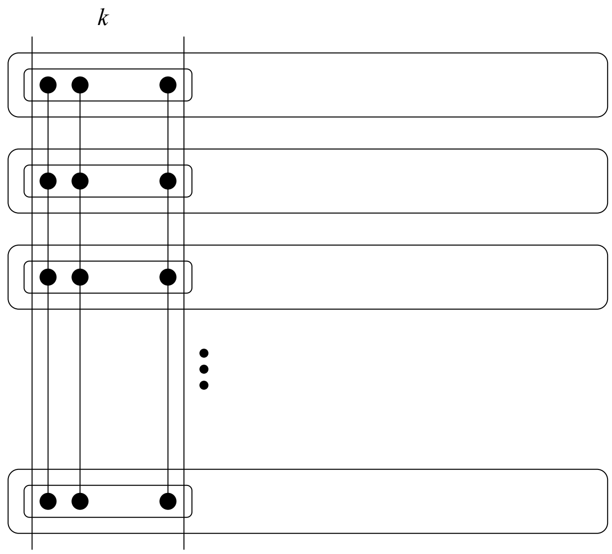
Proof. $\forall r \in RT(s, 1)$, we have r parallel classes in $RT(s, 1)$. Let $\{\infty_1, \infty_2, \cdots, \infty_r\}$ be a new group and attach ∞_i to the i-th parallel class, we obtain a $T[r, s + 1; 1]$. Therefore, $RT(s, 1) \subseteq T(s + 1, 1)$. On the other hand, delete one group from a $T[r, s + 1; 1]$ we have an $RT[r, s; 1]$. This concludes the proof. ∎

Lemma 3.3.7. $RT(s, \lambda) \subseteq T(s + 1, \lambda)$.

Proof. Instead of one parallel class, we attach one ∞_i to λ parallel classes obtained in $RT(s,\lambda)$. Note here that two distinct ∞_i's must attach to two mutually disjoint "λ" parallel classes. ∎

Theorem 3.3.8. *If $r \in B(K,1)$, and $K \subseteq RT(s,1)$, then $r \in T(s,1)$.*

($B(K,1) = \{v \mid$ a $(v,K,1)$-design exists $\}$.)
Proof. For each block B of size k, define an $RT[k,s;1]$ with a vertical parallel class. Then, delete all vertical parallel classes and replace them with a collection of r vertical blocks. Hence we have $r \in T(s,1)$. ∎



Corollary 3.3.9. *If $r \in B(K,1)$ and $K \subseteq T(s+1,1)$, then $r \in T(s,1)$.*

Proof. By Lemma 3.3.6. ∎

Theorem 3.3.10. *If q is a prime power, then $q \in T(q+1,1)$.*

Proof. It is well-known that if q is a prime power then there exist $q-1$ mutually orthogonal latin squares of order q, let them be $L^{(1)}, L^{(2)}, \cdots, L^{(q-1)}$. Also, let $L^{(r)}$ and $L^{(c)}$ be two arrays defined as follows:

$L^{(r)}$:

0	0	$\cdots$	0
1	1	$\cdots$	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$q-1$	$q-1$	$\cdots$	$q-1$

$L^{(c)}$:

0	1	$\cdots$	$q-1$
0	1	$\cdots$	$q-1$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	1	$\cdots$	$q-1$

Clearly, $L^{(r)}, L^{(c)}, L^{(1)}, \cdots, L^{(q-1)}$ are $q+1$ mutually orthogonal arrays. Now, we are ready to define a $T[q,q+1;1]$. For $i \in \{r,c\} \cup \mathbb{Z}_q$, let $G_i = \{(i,j) \mid j \in \mathbb{Z}_q\}$ and $B_{h,k}$ be a block (corresponding to the (h,k)-entry of the above $q+1$ arrays) such that

$$B_{h,k} = \{(r,h),(c,k),(0,L^{(0)}_{h,k}),(1,L^{(1)}_{h,k}),\cdots,(q-1,L^{(q-1)}_{h,k})\}.$$

It is left to check the above design is a $T[q,q+1;1]$. Let (i_1,j_1) and (i_2,j_2) be two elements in $\bigcup_{i\in\{r,c\}\cup\mathbb{Z}_q} G_i$ such that $i_1 \neq i_2$. Since the i_1-th array and i_2-th array are orthogonal, there exist an entry (h,k) such that $L^{i_1}_{h,k} = j_1$ and $L^{i_2}_{h,k} = j_2$. This implies that (i_1,j_1) and (i_2,j_2) are in the unique block $B_{h,k}$. ∎

Remark. In order to obtain more designs with "good" properties we shall introduce a new class called "truncated transversal designs."

3.4 Truncated Transversal Designs

A group divisible design $GD[n,m;k;\lambda]$ can be extended to a more general type by allowing (1) groups of distinct sizes and (2) blocks of distinct sizes. Therefore, if M is the set of distinct sizes of groups and K is the set of distinct block sizes, then we have a pairwise balanced GDD, $PGD[M,m;K;\lambda]$. In fact, if we let $\tilde{K} = M \cup K$, then we can define a PBD with block sizes in $\tilde{K}$. For example, a $GD[n,m;k;\lambda]$ can be extended to a $B[nm;K;\lambda]$ where $K = \{n,k\}$, i.e., we have $nm \in B(K,\lambda)$ where $K = \{n,k\}$. So, starting from a known transversal design, we may construct PBD's of distinct orders with a suitable set of block sizes.

Definition 3.4.1. Let $(X,\mathcal{G},\mathbb{B})$ be a $T[r,s+t;\lambda]$ where $X = \bigcup_{i=1}^{s+t} G_i$ and $\mathcal{G} = \{G_i \mid i = 1,2,\cdots,s+t\}$. A design $(X\backslash S,\mathcal{G}',\mathbb{B}')$ is a truncated transversal design if

(1) $S = \bigcup_{i=1}^{t} S_i$;

(2) $G'_{s+t} = G_{s+t}\backslash S_i$, $i \in \{1,2,\cdots,t\}$; $G'_j = G_j$, $j \in \{1,2,\cdots,s\}$;

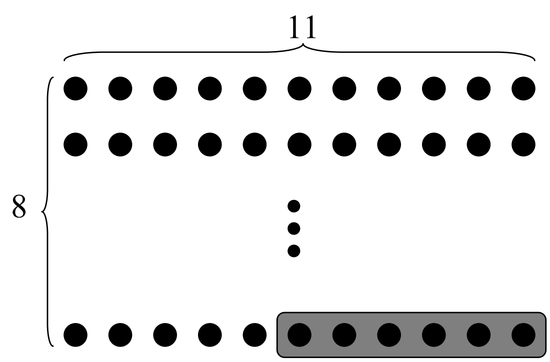
(3) $\mathbb{B}' = \{B\backslash S \mid B \in \mathbb{B}\}$.

Lemma 3.4.2. *If $(X\backslash S,\mathcal{G}',\mathbb{B}')$ is defined as above, and $|G'_j| = r$ for $j = 1,2,\cdots,s$ and $|G'_{s+i}| = r_i$, then we have a $(|X|-|S|,K,\lambda)$-PBD where $K = \{r,r_1,\cdots,r_t,s+t,s+t-1,\cdots,s\}$.*

Proof. The result follows by letting each group (after truncated) be a block which occurs λ times. ∎

Now, we can apply Lemma 3.3.9and 3.4.2 to construct mutually orthogonal latin squares of "suitable" orders.

Example 3.4.3. $MOLS(82)$, $MOLS(86)$



By using a $T[11,8;1]$ we delete 6 vertices from the last group to obtain a $(82,K,1)$-PBD such that $K = \{5,7,8,11\}$. Since for each element x in K we have at least 4 mutually orthogonal latin squares of order x, we have a $82 \in T(5,1)$, i.e., we have 3 $MOLS(82)$'s. For 86, we delete two vertices from the last group and thus $K = \{7, 8, 9, 11\}$. This implies that $86 \in T(7,1)$, i.e., there are at least 5 $MOLS(86)$'s.

Example 3.4.4. $MOLS(90)$, $MOLS(94)$

We start with a $T[11, 9; 1]$ and delete 6 and 3 vertices respectively from the last two groups. This implies that we have $K = \{5, 7, 8, 9, 11\}$ and thus $90 \in T(5,1)$. For this other case, we delete 3 and 2 vertices respectively from the last two groups. Hence $K = \{7, 8, 9, 11\}$ and $94 \in T(7,1)$.

By the above two examples, there are evidence that Euler's conjecture on MOLS is false. It is worth of mention here that in fact we can prove that there are at least two $MOLS(v)$'s provided $v \in \{2, 6\}$ and three $MOLS(v)$'s provided $v \geq 11$.

Exercise 3.4.5. Find two $MOLS(22)$'s by using a $(22, \{4, 7\}, 1)$-PBD.

Exercise 3.4.6. Find three $MOLS(21)$'s.

Chapter 4

Steiner Quadruple Systems

4.1 General properties of t-designs

Review that a t-(v, k, λ) design $(V, \mathbb{B})$ is a collection $\mathbb{B}$ of blocks of size k defined on a v-set V such that every t-subset of V occurs together in exactly λ blocks of $\mathbb{B}$. Directly from the definition, we have the following results.

Proposition 4.1.1. If $(V, \mathbb{B})$ is a t-(v, k, λ) design and $x \in V$, then $(V \setminus \{x\}, \mathbb{B}')$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design where $\mathbb{B}' = \{B \setminus \{x\} \mid B \in \mathbb{B} \text{ and } x \in V\}$.

Proof. Let T be a set of $t-1$ elements in $V \setminus \{x\}$. Since $T \cup \{x\}$ is a t-subset of V, and $(V, \mathbb{B})$ is a t-(v, k, λ) design, $T \cup \{x\}$ occurs in exactly λ blocks of $\mathbb{B}$. Hence T occurs in exactly λ blocks of $\mathbb{B}'$. This implies that $(V \setminus \{x\}, \mathbb{B}')$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design. ∎

More generally, we have the followings.

Proposition 4.1.2. If $(V, \mathbb{B})$ is a t-$(v, k, 1)$ design and S is a subset of V such that $s = |S| < t$, then $(V \setminus S, \mathbb{B}')$ is a $(t-s)$-$(v-s, k-s, 1)$ design where $\mathbb{B}' = \{B \setminus S \mid B \in \mathbb{B}$ and $S \subseteq B\}$.

Remark that the above result is true only when $\lambda = 1$.

Proposition 4.1.3. If $(V, \mathbb{B})$ is a t-(v, k, λ) design and $1 \le s < t$, then $(V, \mathbb{B})$ is an s-(v, k, λ') design where $\lambda' = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}$.

Proof. Let S be a subset of V such that $|S| = s$. Then, there are $\binom{v-s}{t-s}$ ways to choose $t-s$ elements from $V \setminus S$ to form a t subset of V. Since $(V, \mathbb{B})$ is a t-(v, k, λ) design, there are clearly $\lambda \binom{v-s}{t-s}$ ways to obtain such t-subsets. On the other hand, each block containing S can provide $\binom{k-s}{t-s}$ $(t-s)$-subsets. Hence, $\lambda' = \lambda \binom{v-s}{t-s} / \binom{k-s}{t-s}$. ∎

Example 4.1.4. A 3-$(10, 4, 1)$ design is also a 2-$(10, 4, 4)$ design.

Sol. $\binom{10-2}{3-2} / \binom{4-2}{3-2} = 4$.

Definition 4.1.5. A t-$(v, t+1, 1)$ design is referred to as a Steiner t-design and a Steiner quadruple system when $t = 3$ (Steiner triple system when $t = 2$).

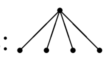
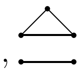
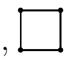
4.2 Steiner quadruple systems (SQS)

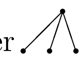
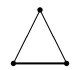
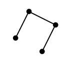
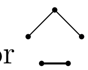
By Proposition 4.1.1, it is easy to see that if $(V, \mathbb{B})$ is a Steiner quadruple system of order v, then $(V \setminus \{x\}, \mathbb{B}')$ is a Steiner triple system of order $v-1$ where $x \in V$, and $\mathbb{B}' = \{B \setminus \{x\} \mid x \in B \text{ and } B \in \mathbb{B}\}$. Hence, we have

Proposition 4.2.1. If an $SQS(v)$ exists, then $v \equiv 2$ or $4 \pmod 6$.

Example 4.2.2. For each $n \geq 2$, an $SQS(2^n)$ exists.

Sol. Let $V = \mathbb{Z}_2^n$ and $\mathbb{B} = \big\{\{\vec{x}, \vec{y}, \vec{z}, \vec{w}\} \mid \vec{x}, \vec{y}, \vec{z}, \vec{w} \in \mathbb{Z}_2^n \text{ and } \vec{x} + \vec{y} + \vec{z} + \vec{w} = \vec{0}\big\}$. Then, it is a routine matter to check $(V, \mathbb{B})$ is an $SQS(2^n)$.

Example 4.2.3. Let $V = E(K_5)$ and $\mathbb{B}$ consists all blocks of the forms: , , . Then $(V, \mathbb{B})$ is an $SQS(10)$.

Sol. For any three edges of K_5, either  or  or  or  occur. Therefore, in each case, it is contained in a block we define. The proof follows by checking the number of blocks, there are 30 blocks.

Example 4.2.4. Let $V = \mathbb{Z}_7 \times \mathbb{Z}_2$, $B_1 = \{(0,0), (3,0), (2,1), (6,1)\}$, $B_2 = \{(0,0), (3,0), (1,1), (5,1)\}$, $B_3 = \{(0,0), (3,0), (2,1), (6,1)\}$, $B_4 = \{(0,0), (3,0), (2,0), (4,1)\}$, and $B_5 = \{(0,0), (3,0), (4,0), (5,0)\}$. Moreover, let $\pi_1((x,a)) = (x+1, a)$, $\pi_2((x,a)) = (2x, a)$ and $\pi_3((x,a)) = (-x, a+1)$. Then $(V, \mathbb{B})$ is an $SQS(14)$ where $\mathbb{B} = \bigcup_{i=1}^{5} \mathbb{B}_i$ and $\mathbb{B}_i$ contains the block B_i and all the blocks B where $B = \pi_j(B')$ whenever $B' \in \mathbb{B}_i$.

Sol. Check that $|\mathbb{B}_1| = |\mathbb{B}_2| = |\mathbb{B}_3| = 21$ and $|\mathbb{B}_4| = |\mathbb{B}_5| = 14$.

Theorem 4.2.5. *If an $SQS(v)$ exists, then an $SQS(2v)$ exists.*

Proof. This construction is known as a doubling construction.

Method 1

Let X and Y be two disjoint v-sets such that $(X, \mathbb{B}_1)$ and $(Y, \mathbb{B}_2)$ are two $SQS(v)$'s. Let $\mathcal{G}$ and $\mathcal{F}$ be two 1-factorizations of K_v defined on X and Y respectively. So, $\mathcal{G} = \{G_1, G_2, \cdots, G_{v-1}\}$ and $\mathcal{F} = \{F_1, F_2, \cdots, F_{v-1}\}$ where G_i's and F_j's are 1-factors of the complete graphs of order v defined on X and Y respectively. Now, let $\langle G_i, F_i \rangle = \big\{\{a, b, c, d\} \mid ab \in E(G_i) \text{ and } cd \in E(F_i)\big\}$. It is not difficult to see that $\langle G_i, F_i \rangle$ contains $\frac{v^2}{4}$ 4-subsets for each $i \in \{1, 2, \cdots, v-1\}$. Now, let $\mathbb{B}$ be obtained by $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \left(\bigcup_{i=1}^{v-1} \langle G_i, F_i \rangle\right)$. Then, $(X \cup Y, \mathbb{B})$ is an $SQS(2v)$.

Method 2

Let X and X' be two disjoint v-sets such that $x \in X$ if and only if $x' \in X$. Moreover, let $(X, \mathbb{B})$ be an $SQS(v)$. Now, we define $\tilde{\mathbb{B}}$ on $X \cup X'$ as follows:

(1) $\{x, y, z, w'\}$,$\{x, y, z', w\}$,$\{x, y', z, w\}$,$\{x', y, z, w\}$,$\{x', y', z', w\}$,$\{x', y', z, w'\}$,$\{x', y, z', w'\}$ and $\{x, y', z', w'\}$ are blocks in $\tilde{\mathbb{B}}$ if $\{x, y, z, w\} \in \mathbb{B}$.

(2) $\{x, y, x', y'\} \in \tilde{\mathbb{B}}$ for any pair of distinct elements x and y in X.

Therefore, there are $8 \cdot |\mathbb{B}| + \binom{v}{2}$ blocks in $\tilde{\mathbb{B}}$. This implies that $|\tilde{\mathbb{B}}| = \frac{2v(2v-1)(2v-2)}{24}$. By the fact that any three distinct elements of $X \cup X'$ occur together in one block of $\tilde{\mathbb{B}}$, we conclude the construction. ∎

Note. A design $(X, \mathbb{B})$ is k-chromatic if there exists a mapping $\phi : X \to \{1, 2, \cdots, k\}$ such that for each $B \in \mathbb{B}$, $\{1, 2, \cdots, k\} \subseteq \phi(B)$.

Note. The $SQS(2v)$ obtained by Method 2 is 2-chromatic.

Problem. Prove or disprove that there exists a 2-chromatic $SQS(v)$ for each $v \equiv 2$ or $4 \pmod 6$.

Example 4.2.6. A 2-chromatic $SQS(10)$.

Let $\phi : E(K_5) \to \{1, 2\}$ such that every edge of one 5-cycle of K_5 receives color 1 and the edges of the other 5-cycle receive color 2. Then, every block defined in Example 4.2.3 is 2-colored.

Example 4.2.7. For each $v \equiv 4$ or $8 \pmod{12}$, there exists a 2-chromatic $SQS(v)$.

Sol. Since an $SQS(\frac{v}{2})$ exists, by Method 2, we have a 2-chromatic $SQS(v)$.

Theorem 4.2.8. *An $SQS(v)$ exists if and only if $v \equiv 2$ or $4 \pmod 6$.*

Proof. (By H. Hanani, 1960)

The proof follows by the following 5 constructions: (1) $v \to 2v$; (2) $v \to 3v - 2$; (3) for $v \equiv 2 \pmod{12}$, $v \to 3v - 8$; (4) for $v \equiv 10 \pmod{12}$, $v \to 3v - 4$; and (5) $v \to 6v - 10$. (Note that small orders can be obtained directly.) ∎

(∗) (2)∼(5) can be replaced by the following construction which was obtained by Hartman.

Theorem 4.2.9. *If an $SQS(v)$ contains an $SQS(u)$ as a proper subsystem, then there exists an $SQS(3v - 2u)$.*

Proof. See reference. (A. Hartman, "A general recursion construction for quadruple systems", *JCT(A)*, **33**(1982), 121–134.)

For understanding the tripling constructions, we present the following construction.

$v \to 3v - 2$ Construction (An $SQS(3v - 2)$ contains an $SQS(v)$.)

Let $V = (\mathbb{Z}_{v-1} \times \mathbb{Z}_3) \cup \{\infty\}$ and the quadruples of $\mathbb{B}$ are of the following three types. (Let the $SQS(v)$ be $(\mathbb{Z}_v, \mathbb{B}')$.)

Type 1: For each block $B = \{0, x, y, z\}$ define an $SQS(10)$ $(\mathbb{B}'(B))$ on $(\{x, y, z\} \times \mathbb{Z}_3) \cup \{\infty\}$ which contains the following four blocks $B_1 = \{\infty, (x, 0), (x, 1), (x, 2)\}$, $B_2 = \{\infty, (y, 0), (y, 1), (y, 2)\}$, $B_3 = \{\infty, (z, 0), (z, 1), (z, 2)\}$, $B_4 = \{\infty, (x, 0), (y, 0), (z, 0)\}$. Type 1 blocks are in $\mathbb{B}'(B) \setminus \{B_1, B_2, B_3\}$.

Type 2: For each block $B = \{w, x, y, z\} \in \mathbb{B}'$ which does not contain 0, let $\{(w, i), (x, j), (y, k), (z, l)\} \in \mathbb{B}$ for all $i, j, k, l \in \mathbb{Z}_3$ such that $i + j + k + l \equiv 0 \pmod 3$.

Type 3: For each $x \in \mathbb{Z}_{v-1}$, let $\{\infty, (x, 0), (x, 1), (x, 2)\} \in \mathbb{B}$.

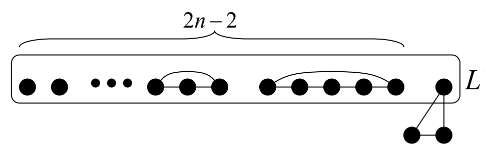
Proof. There are $27 \times \frac{v(v-1)(v-2)}{24} + (v - 1) = \frac{3v(3v-3)(3v-6)+24(v-1)}{24} = \frac{(3v-3)[3v(3v-6)+8]}{24} = \frac{(3v-2)(3v-3)(3v-4)}{24}$ blocks. Now, it is left to check for any three distinct elements in $(\mathbb{Z}_{v-1} \times \mathbb{Z}_3) \cup \{\infty\}$ there exists at least one block (defined above) which contains them. We leave this job to the readers. ∎

Exercise. Construct an $SQS(26)$.

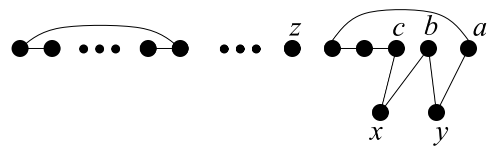
4.3 4-cycle system of K_{2n+1} with 2-regular leave

Theorem 4.3.1. *Let L be a 2-regular subgraph of K_{2n+1} such that $K_{2n+1} - L$ of size a multiple of 4. Then $C_4 \mid K_{2n+1} - L$.*

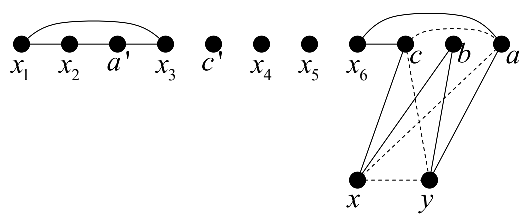
Proof. By induction on n and it is true for $n = 1$ and 2 by direct checking. Assume that the assertion is true for $n - 1$ and let $G = K_{2n+1} - L$. Notice that $\binom{2n+1}{2} - \binom{2n-1}{2} \equiv 3 \pmod 4$. Therefore, if L contains a triangle, then we can arrange L as in the following figure and obtain an L' for K_{2n-1}. Since $\binom{2n+1}{2} - |L|$ is a multiple of 4, $\binom{2n-1}{2} - |L'|$ is also a multiple of 4. By induction $K_{2n-1} - L'$ can be decomposed into 4-cycles. Then, $K_{2n+1} - L$ can be decomposed into 4-cycles by combining $C_4 \mid K_{2n-1} - L'$ and $C_4 \mid K_{2,2n-2}$.



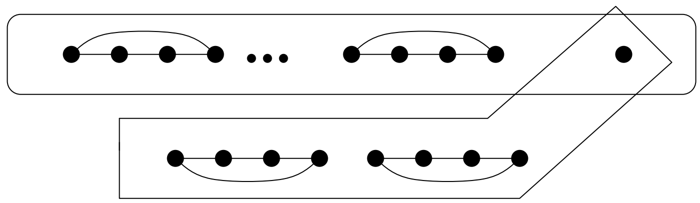
On the other hand, if L does not contain a triangle. First, if L is an empty graph, then $4 \mid \binom{2n+1}{2}$ and thus $n \equiv 0 \pmod 4$. The proof follows by the existence of 4-cycle systems of orders $8k + 1$, $k \in \mathbb{N}$. Now, assume that L contains a cycle of length larger than 3, let C_t be the one with length $t > 5$ (if exists). For convenience, L is arranged as follows:



Now, by deleting xc, xb, yb, ya and adding ca, we have a new leave L' of K_{2n-1}. Since $|L'| = |L| - 3$, $K_{2n-1} - L'$ can be decomposed into 4-cycles. It's left to decompose the subgraph left, see the following example $K_{13} - L$ for the idea and we omit the details of writing.

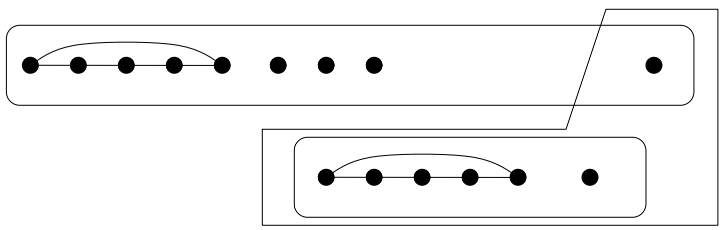


In the C_4-decomposition of $K_{11} - (C_4 \cup C_3)$, ca is not included in any 4-cycles. The 4-cycles left are (x, y, c, a) and C_4's in $K_{2,8}$. On the other hand, if there are two cycles of length 4 and $4\big||L|$, then the graph looks as follows. ($2n+1$ is congruent to 1 modulo 8).



So, we have the case "L contains only C_4's and C_5's" left. First, if L contains only 4-cycles, then the proof follows by using $9 \longrightarrow 2n+1$ (?). See the following figure.

Finally, L contains at most one 4-cycle and a bunch of 5-cycles. First, if $|L| < 2n+1$, then the proof follows by using a $7 \longrightarrow 2n+1$ construction, see the figure below. This is by the fact $C_4 \mid K_7 - C_5$.



So, it is left to consider the case $|L| = 2n+1$.

(1) L contains a C_4.

It is not difficult to check that in this case $2n+1 \equiv 19 \pmod{40}$. By direct construction,we have $C_4 \mid K_{19} - (C_4 \cup C_5 \cup C_5 \cup C_5)$. Also, $C_4 \mid K_{2n-17} - (8k)C_5$ where $2n+1 = 40k+19$. The proof follows by adding the set of 4-cycles from $K_{18,40k}$.

(2) L contains only C_5's.

Now, $2n+1 \equiv 35 \pmod{40}$. For $C_4 \mid K_{35} - 7C_5$, we may use difference method to obtain the construction, namely difference 7 for seven 5-cycles and $\{1, 2, 3, 4\}$, $\{5, 6, 8, 9\}$, $\{10, 11, 12, 13\}$, $\{14, 15, 16, 17\}$ for cyclic 4-cycles. By the same technique used above, we have the construction. ∎

In fact, we can prove a stronger result which is close related to packing, namely, packing $K_{2n+1} - L$ into 4-cycles with leaves: cycles.

Theorem 4.3.2. *Let L be a 2-regular subgraph of K_{2n+1}. Then $K_{2n+1} - L$ can be decomposed into (1) 4-cycles if $\binom{2n+1}{2} - |L|$ is a multiple of 4; (2) 4-cycles and one 5-cycle if $\binom{2n+1}{2} - |L| \equiv 1 \pmod 4$; (3) 4-cycles and one 6-cycle if $\binom{2n+1}{2} - |L| \equiv 2 \pmod 4$; and (4) 4-cycles and one 3-cycle if $\binom{2n+1}{2} - |L| \equiv 3 \pmod 4$.*

Proof. by induction on n and it is clear that the assertion is true for $n = 1$ and 2. Assume that the assertion is true for $n-1$ and consider $K_{2n+1} - L$. Then we may follow a similar argument as we used in the proof of Theorem 4.3.1. To start, we check if L has a C_3 and then a cycle of length larger than 4. Finally, consider all components of L are 4-cycles. ∎

We may also consider the packing of K_{2n} with 4-cycles and we have the following result.

Theorem 4.3.3. *Let F be a spanning odd forest of K_{2n} such that $\binom{2n}{2} - \|F\|$ is a multiple of 4. Then, $K_{2n} - F$ can be decomposed into 4-cycles.*

The proof of Theorem 4.3.3 is also by induction on n except it is a little bit more complicate, see the following reference: Hung-Lin Fu and Chris Fodger, "Forest leaves and 4-cycles", *JGT*, **33**, (2000), 161–166.

A surprising fact is that the work of Theorem 4.3.3 can not be extended to C_3's with ease. So far, the problem for 3-cycle packing is still unsolved. A more general problem to consider cycle packings is of the following form:

Problem: Determine $f(n)$ such that if a graph G of order n with $\delta(G) \geq f(n)$, then G can be decomposed into k-cycles provided G is k-sufficient.

Note For odd n, we know that for $k = 3$ or 4 or 6, $f(n) \leq n - 3$.

Concluding Remark

From what has happened over the years, it is proved that combinatorial designs play a key role in many aspects of applications. Not only on experimental designs including group testing but also in Coding Theory and Cryptography. Designing efficient experiments is important that is for sure. But, constructing better codes is even more applicable in real world. Don't mention those designs used in keeping the security of transmitting data. Therefore, learning more topics in combinatorial designs will be a great aid to the study of discrete models.