Theorem (Fisher's inequality) Exercise 2

For a 2-(v,k,λ) design with b blocks and $v>k$, we have $b \geq v$.

Pf: Let $(X, \mathcal{B})$ be a 2-(v,k,λ) design with incidence matrix $M=[m_{ij}]$.
For $1 \leq i \leq v$, define $S_i = (m_{i1}, m_{i2}, \ldots, m_{ib})$ to be the i-th row vector of M.
Then it suffices to show that $S_1, S_2, \ldots, S_v$ are linearly independent over $\mathbb{R}$.
Suppose not. Then there exist $c_1, c_2, \ldots, c_v \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^{v} c_i S_i = 0$.
Thus

$$0 = \left\langle \left(\sum_{i=1}^{v} c_i S_i\right), \left(\sum_{j=1}^{v} c_j S_j\right) \right\rangle \quad \text{内積}$$

$$= \sum_{i=1}^{v} c_i^2 \langle S_i, S_i \rangle + \sum_{i \neq j} c_i c_j \langle S_i, S_j \rangle$$

$$= \sum_{i=1}^{v} c_i^2 \cdot r + \sum_{i \neq j} c_i c_j \cdot \lambda \quad \left(\begin{array}{l} \langle S_i, S_i \rangle = \#\text{ of blocks containing the } i\text{-th element } x_i, \\ \langle S_i, S_j \rangle = \#\text{ of blocks containing } x_i \text{ and } x_j. \end{array} \right)$$

$$= \sum_{i=1}^{v} c_i^2 (r-\lambda) + \lambda \left(\sum_{i=1}^{v} c_i\right)^2 > 0 \quad (\text{since } r > \lambda), \text{ a contradiction.} \quad \blacksquare$$

by the fact that $\lambda(v-1) = r(k-1)$ and $v>k$

Good!

李 [illegible]

Note rank$(M) \leq \min\{v, b\}$.