Theorems in Graph Theory

Theorem 1 (Degree-Sum Formula). *For any graph G,*

$$\sum_{v \in V} d(v) = 2\|G\|.$$

Theorem 2. *In a graph G, the average vertex degree is $\dfrac{2\|G\|}{|G|}$, and hence*

$$\delta(G) \leq \frac{2\|G\|}{|G|} \leq \Delta(G).$$

Theorem 3. *In any graph, the number of vertices of odd degree is even.*

Theorem 4. *A k-regular graph with n vertices has $\frac{nk}{2}$ edges.*

Theorem 5. *If $k > 0$, then a k-regular bipartite graph has the same number of vertices in each partite set.*

Theorem 6. *For a simple graph G with vertices $v_1, ..., v_n$ and $n \geq 3$,*

$$e(G) = \frac{\sum_{i=1}^{n} e(G - v_i)}{n-2} \quad \textit{and} \quad d(v_j) = \frac{\sum_{i=1}^{n} e(G - v_i)}{n-2} - e(G - v_j).$$

Theorem 7. *Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.*

Theorem 8. *For every graph G and every integer $r \geq \Delta(G)$, there exists an r-regular graph H containing G as an induced subgraph.*

Theorem 9. *A sequence $d_1, d_2, ..., d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$, $n \geq 2$, $d_1 \geq 1$, is graphical if and only if the sequence $d_2 - 1, d_3 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n$ is graphical.*

Theorem 10. *A sequence* $d_1, d_2, ..., d_n$ $(n \geq 2)$ *of nonnegative integers with* $d_1 \geq d_2 \geq \cdots \geq d_n$ *is graphical if and only if* $\sum_{i=1}^{n} d_i$ *is even and for each integer* k*,* $1 \leq k \leq n-1$*,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Theorem 11. *Every* $u - v$ *walk in a graph contains a* $u - v$ *path.*

Theorem 12. *Every graph with* n *vertices and* k *edges has at least* $n - k$ *components.*

Theorem 13. *If* G *is a simple graph of order* n *with* $\delta(G) \geq \frac{n-1}{2}$*, then* G *is connected.*

Theorem 14. *Every loopless graph* G *has a bipartite subgraph with at least* $\frac{e(G)}{2}$ *edges.*

Theorem 15. *If* A *is the adjacency matrix of a graph* G *with* $V(G) = \{v_1, v_2, ..., v_n\}$*, then the* (i, j) *entry of* A^k*,* $k \geq 1$*, is the number of different* $v_i - v_j$ *walks of length* k *in* G*.*

Theorem 16 (König 1936)**.** *A graph is bipartite if and only if it has no odd cycle.*

Theorem 17. *Every loopless graph* G *contains a spanning bipartite subgraph* F *such that* $d_F(v) \geq \frac{1}{2}d_G(v)$ *for all* $v \in V$*.*

Theorem 18. *Every graph with average degree at least* $2k$*, where* k *is a positive integer, has an induced subgraph with minimum degree at least* $k+1$*.*

Theorem 19. *If* H *is a subgraph of* G*, then* $d_G(u,v) \leq d_H(u,v)$*.*

Theorem 20. *For every connected graph* G*,*

$$rad(G) \leq diam(G) \leq 2\,rad(G).$$

Theorem 21. *If* G *is a simple graph, then* $diam(G) \geq 3 \Rightarrow diam(\overline{G}) \leq 3$*.*

Theorem 22. *The center of a tree is a vertex or an edge.*

Theorem 23. *Every graph is the center of some connected graph.*

Theorem 24. *A graph* G *is the periphery of some connected graph if and only if every vertex of* G *has eccentricity* 1 *or no vertex of* G *has eccentricity* 1*.*

Theorem 25. *If* D *is a digraph of order* n *and size* m *with* $V(D) = \{v_1, v_2, ..., v_n\}$*, then*

$$\sum_{i=1}^{n} d^+(v) = \sum_{i=1}^{n} d^-(v) = m.$$

Theorem 26. *A vertex* v *of a connected graph* G *is a cut-vertex of* G *if and only if there exist vertices* u *and* w $(u, w \neq v)$ *such that* v *is on every* $u-w$ *path of* G*.*

Theorem 27. *Every nontrivial connected graph contains at least two vertices that are not cut-vertices.*

Theorem 28. *An edge e of a connected graph G is a bridge of G if and only if there exist vertices u and w such that e is on every $u - w$ path of G.*

Theorem 29. *An edge e of a graph G is a bridge (or a cut-edge) of G if and only if e lies on no cycle of G.*

Theorem 30. *Every closed odd walk contains an odd cycle.*

Theorem 31. *The complete graph K_n can be expressed as the union of k-bipartite graphs if and only if $n \leq 2^k$.*

Theorem 32. *If we decompose a complete graph K_n, into m cliques different from K_n, such that evely edge is in a unique clique, then $m \geq n$.*

Theorem 33 (Graham–Pollak 1971)**.** *If K_n is decomposed into complete bipartite subgraphs $H_1, ..., H_m$, then $m \geq n - 1$.*

Theorem 34. *For a connected nontrivial graph with exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$.*

Theorem 35. *A graph G of order at least 3 has no cut-vertices if and only if every two vertices of G lie on a common cycle of G.*

Theorem 36 (Whitney 1932)**.** *A graph G of order at least 3 is 2-connected if and only if there exist two internally disjoint $u - v$ paths for every two distinct vertices u and v of G.*

Theorem 37 (Expansion Lemma)**.** *If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.*

Theorem 38. *If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected.*

Theorem 39. *Let G be a connected graph with one or more cut-vertices. Then among the blocks of G, there are at least two which contain exactly one cut-vertex of G.*

Theorem 40. *The center of every connected graph G lies in a single block of G.*

Theorem 41. *If G is a critical block of order at least 4, then G contains a vertex of degree 2.*

Theorem 42. *If G is a minimal block of order at least 4, then G contains a vertex of degree 2.*

Theorem 43. *Every edge-transitive graph without isolated vertices is either vertex-transitive or bipartite.*

Theorem 44. *Every regular graph of order at least 3 is reconstructible.*

Theorem 45. *Disconnected graphs of order at least 3 are reconstructible.*

Theorem 46. *Every nontrivial tree has at least two end-vertices.*

Theorem 47. *A graph G of order n and size m is a tree if and only if G is acyclic and $n = m + 1$.*

Theorem 48. *A graph G of order n and size m is a tree if and only if G is connected and $n = m + 1$.*

Theorem 49. *A graph G is a tree if and only if every two distinct vertices of G are connected by a unique path of G.*

Theorem 50. *A sequence $d_1, d_2, ..., d_n$ of $n \geq 2$ positive integers is the degree sequence of a tree of order n if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.*

Theorem 51. *Let T be a nontrivial tree with $\Delta(T) = k$, and let n_i be the number of vertices of degree i for $i = 1, 2, ..., k$. Then*

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2.$$

Theorem 52 (Exchange Property)**.** *If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.*

Theorem 53. *If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.*

Theorem 54 (Hidden Tree)**.** *Let T be a tree of order k, and let G be a graph with $\delta(G) \geq k - 1$. Then T is a subgraph of G.*

Theorem 55. *Let n and k be positive integers with $n \geq 2k$. Then every graph G of order n with $\delta(G) \geq k$ contains every forest of size k without isolated vertices as a subgraph.*

Theorem 56 (Cayley's Formula 1889)**.** *There are n^{n-2} distinct labeled trees of order n.*

Theorem 57. *Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then $\tau(G) = \tau(G - e) + \tau(G \cdot e)$.*

Theorem 58 (Matrix-Tree Theorem)**.** *If G is a nontrivial labeled graph with adjacency matrix A and degree matrix D, then the number of distinct spanning trees of G is the value of any cofactor of the matrix $D - A$.*

Theorem 59 (Rosa 1967)**.** *If a tree with m edges has a graceful labeling, then K_{2m+1} has a decomposition into $2m + 1$ copies of T.*

Theorem 60. *Let G be a connected graph of order $n \geq 3$ that is not complete. For each edge-cut X of G, there is a vertex-cut U of G such that $|U| \leq |X|$.*

Theorem 61 (Whitney 1932)**.** *For every graph G,*

$$\kappa(G) \leq \kappa_1(G) \leq \delta(G).$$

Theorem 62. *If G is a 3-regular graph, then $\kappa(G) = \kappa_1(G)$.*

Theorem 63. *If S is a set of vertices in a graph G, then*

$$|[S, \overline{S}]| = [\sum_{v \in S} d(v)] - 2e(G[S]).$$

Theorem 64. *If G is a simple graph and $|[S, \overline{S}]| < \delta(G)$ for some nonempty proper subset S of $V(G)$, then $|S| > \delta(G)$.*

Theorem 65. *If G is a connected graph, then an edge cut F is a bond if and only if $G - F$ has exactly two components.*

Theorem 66. *Let G be a graph of order $n \geq 2$, and let k be an integer such that $1 \leq k \leq n - 1$. If*

$$d(v) \geq \lceil \frac{n + k - 2}{2} \rceil$$

for every vertex v of G, then G is k-connected.

Theorem 67. *A nontrivial graph G is k-edge-connected if and only if there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G) - W$ is less than k.*

Theorem 68 (Plesnik)**.** *If G is a graph of diameter 2, then $\kappa_1(G) = \delta(G)$.*

Theorem 69. *If G is a graph of order $n \geq 2$ such that for all distinct nonadjacent vertices u and v,*

$$d(u) + d(v) \geq n - 1,$$

then $\kappa_1(G) = \delta(G)$.

Theorem 70. *Deletion of an edge reduces connectivity by at most 1.*

Theorem 71 (Menger's Theorem, Menger 1927). *Let u and v be nonadjacent vertices in a graph G. Then the minimum number of vertices that separate u and v is equal to the maximum number of internally disjoint $u - v$ paths in G.*

Theorem 72 (Whitney). *A nontrivial graph G is k-connected if and only if for each pair u, v of distinct vertices there are at least k internally disjoint $u - v$ paths in G.*

Theorem 73 (Fan Lemma, Dirac 1960). *If G is a k-connected graph and $v, v_1, v_2, ..., v_k$ are $k + 1$ distinct vertices of G, then there exist internally disjoint $v - v_i$ paths $(1 \leq i \leq k)$.*

Theorem 74 (Dirac 1960). *Let G be a k-connected graph, $k \geq 2$. Then every k vertices of G lie on a common cycle of G.*

Theorem 75. *If u and v are distinct vertices of a graph G, then the maximum number of edge-disjoint $u - v$ paths in G equals the minimum number of edges of G that separate u and v.*

Theorem 76. *A nontrivial graph G is k-edge-connected if and only if for every two distinct vertices u and v of G, there exist at least k edge-disjoint $u - v$ paths in G.*

Theorem 77. *Let G be a nontrivial connected graph. Then G is eulerian if and only if every vertex of G is even.*

Theorem 78. *Let G be a nontrivial connected graph. Then G contains an eulerian trail if and only if G has exactly two odd vertices. Furthermore, the trail begins at one of these odd vertices and terminates at the other.*

Theorem 79. *A nontrivial connected graph G is eulerian if and only if every edge of G lies on an odd number of cycles.*

Theorem 80. *Let D be a nontrivial connected digraph. Then D is eulerian if and only if $d^+(v) = d^-(v)$ for every vertex v of D.*

Theorem 81. *Let D be a nontrivial connected digraph. Then D has an eulerian trail if and only if D contains vertices u and v such that*

$$d^+(u) = d^-(u) + 1 \quad \textit{and} \quad d^-(v) = d^+(v) + 1$$

and $d^+(w) = d^-(w)$ for all other vertices w of D. Furthermore, the trail begins at u and ends at v.

Theorem 82. *Let G be a connected graph with $X \subseteq E(G)$. Then G contains an even subgraph H with $X \subseteq E(H)$ if and only if X contains no minimal edge-cut of G having odd cardinality.*

Theorem 83. *If G is a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v,*

$$d(u) + d(v) \geq n,$$

then G is hamiltonian.

Theorem 84 (Ore 1960)**.** *Let u and v be distinct nonadjacent vertices of a graph G of order n such that $d(u) + d(v) \geq n$. Then $G + uv$ is hamiltonian if and only if G is hamiltonian.*

Theorem 85. *If G_1 and G_2 are two graphs obtained from a graph G of order n by recursively joining pairs of nonadjacent vertices whose degree sum is at least n, then $G_1 = G_2$, i.e., the closure of G is well-defined.*

Theorem 86 (Bondy–Chvátal 1976)**.** *A graph is hamiltonian if and only if its closure is hamiltonian.*

Theorem 87. *Let G be a graph with at least three vertices. If $C(G)$ is complete, then G is hamiltonian.*

Theorem 88 (Chvátal 1972)**.** *Let G be a graph of order $n \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no value of $k < n/2$ for which $d_k \leq k$ and $d_{n-k} \leq n - k - 1$, then G is hamiltonian.*

Theorem 89. *If G is a graph of order $n \geq 3$ such that $d(v) \geq n/2$ for every vertex v of G, then G is hamiltonian.*

Theorem 90. *If G is a 2-connected graph of order n such that for all distinct nonadjacent vertices u and v,*

$$|N(u) \bigcup N(v)| \geq \frac{2n-1}{3},$$

then G is hamiltonian.

Theorem 91 (Chvátal–Erdos 1972)**.** *Let G be a graph with at least three vertices. If $\kappa(G) \geq \alpha(G)$ then G is hamiltonian.*

Theorem 92 (Bondy)**.** *Let G be a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v,*

$$d(u) + d(v) \geq n.$$

Then $\kappa(G) \geq \alpha(G)$.

Theorem 93. *If G is a 2-connected graph of order n such that*

$$d(u) + d(v) + d(w) \geq \frac{3n}{2}$$

for every set $\{u, v, w\}$ of three independent vertices of G, then G is hamiltonian.

Theorem 94. *If G is a graph of order n such that for all distinct nonadjacent vertices u and v,*

$$d(u) + d(v) \geq n + 1,$$

then G is hamiltonian-connected.

Theorem 95. *If G is a graph of order n such that $d(v) \geq (n+1)/2$ for every vertex v of G, then G is hamiltonian-connected.*

Theorem 96. *Let G be a graph of order $n \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no value of $k \leq n/2$ for which $d_k \leq k$ and $d_{n-k} \leq n - k$, then G is hamiltonian-connected.*

Theorem 97. *If G is a hamiltonian graph of order n and size m where $m \geq n^2/4$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Theorem 98. *Let G be a graph of order $n \geq 3$ such that for all distinct nonadjacent vertices u and v,*

$$d(u) + d(v) \geq n.$$

Then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.

Theorem 99. *If D is a strong nontrivial digraph of order n such that for every pair u, v of distinct nonadjacent vertices,*

$$d(u) + d(v) \geq 2n - 1,$$

then D is hamiltonian.

Theorem 100. *If D is a nontrivial digraph of order n such that whenever u and v are distinct vertices and $(u, v) \notin E(D)$,*

$$d^+(u) + d^-(v) \geq n,$$

then D is hamiltonian.

Theorem 101. *If D is a strong digraph such that $d(v) \geq n$ for every vertex v of D, then D is hamiltonian.*

Theorem 102. *If D is a digraph such that $d^+(v) \geq n/2$ and $d^-(v) \geq n/2$ for every vertex v of D, then D is hamiltonian.*

Theorem 103. *If* G *is a connected graph, then* G^3 *is hamiltonian-connected.*

Theorem 104. *If* G *is a* 2*-connected graph, then* G^2 *is hamiltonian.*

Theorem 105. *If* G *is a* 2*-connected graph, then* G^2 *is hamiltonian-connected.*

Theorem 106. *Let* G *be a graph of order* n. *If* $(\overline{G})^2 \neq K_n$ *and* $(\overline{G})^2 \neq K_n - e$, *then* G^2 *is hamiltonian-connected.*

Theorem 107. *Let* G *be a graph with* $G \neq P_4$. *Then either* G^2 *or* $(\overline{G})^2$ *is hamiltonian-connected.*

Theorem 108. *Let* G *be a graph of order at least* 3. *Then either* G^2 *or* $(\overline{G})^2$ *is hamiltonian.*

Theorem 109. *If* G *is supereulerian, then* $L(G)$ *is hamiltonian.*

Theorem 110. *A digraph* D *is strong if and only if* D *contains a closed spanning walk.*

Theorem 111 (Robbins 1939)**.** *A nontrivial graph* G *has a strong orientation if and only if* G *is* 2*-edge-connected.*

Theorem 112. *A tournament is transitive if and only if it is acyclic.*

Theorem 113. *For every positive integer* n, *there is exactly one transitive tournament of order* n.

Theorem 114. *For every positive integer* n, *there is exactly one acyclic tournament of order* n.

Theorem 115. *Every nontrivial strong tournament has radius* 2.

Theorem 116. *The center of every nontrivial strong tournament contains at least three vertices.*

Theorem 117. *Every tournament contains a hamiltonian path.*

Theorem 118. *Every transitive tournament contains exactly one hamiltonian path.*

Theorem 119. *Every nontrivial strong tournament is vertex-pancyc1ic.*

Theorem 120. *Every nontrivial strong tournament is pancyc1ic.*

Theorem 121 (Euler's Formula, Euler 1758)**.** *If G is a connected plane graph with n vertices, m edges and r regions, then*

$$n - m + r = 2.$$

Theorem 122. *If G is a maximal planar graph of order n and size m, with $n \geq 3$, then $m = 3n - 6$.*

Theorem 123. *If G is a planar graph of order n and size m, with $n \geq 3$, then $m \leq 3n - 6$.*

Theorem 124. *Every planar graph contains a vertex of degree at most* 5.

Theorem 125. *Let G be a maximal planar graph of order $n \geq 4$, and let n_i denote the number of vertices of degree i in G, for $i = 3, 4, ..., k(= \Delta(G))$. Then*

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + ... + (k - 6)n_k + 12.$$

Theorem 126 (Euler Polyhedron Formula)**.** *If V, E and F are the number of vertices, edges and faces of a polyhedron, then*

$$V - E + F = 2.$$

Theorem 127. *There are exactly five regular polyhedra.*

Theorem 128. *The graphs K_5 and $K_{3,3}$ are nonplanar.*

Theorem 129. *The graphs K_4 and $K_{2,3}$ are planar but not outerplanar.*

Theorem 130. *Every simple outerplanar graph has a vertex of degree at most* 2*.*

Theorem 131. *Every maximal outerplanar graph with at least three vertices is* 2*-connected.*

Theorem 132. *If a graph G contains a subgraph that is a subdivision of either K_5 or $K_{3,3}$, then G is nonplanar.*

Theorem 133. *A graph is planar if and only if each of its blocks is planar.*

Theorem 134 (Kuratowski 1930)**.** *A graph is planar if and only if it contains no subgraph that is a subdivision of either K_5 or $K_{3,3}$.*

Theorem 135. *If F is the edge set of a face in a planar embedding of G, then G has an embedding with F being the edge set of the unbounded face.*

Theorem 136. *Every minimal nonplanar graph is* 2*-connected.*

Theorem 137. *Let $S = \{x, y\}$ be a separating 2-set of G. If G is nonplanar, then adding the edge xy to some S-lobe of G yields a nonplanar graph.*

Theorem 138. *If G is a graph with fewest edges among all nonplanar graphs with Kuratowski subgraphs, then G is 3-connected.*

Theorem 139 (Thomassen 1980)**.** *Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.*

Theorem 140. *If G has no Kuratowski subgraph, then also $G \cdot e$ has no Kuratowski subgraph*

Theorem 141 (Tutte 1960, 1963)**.** *If G is a 3-connected graph with no subdivision of either K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.*

Theorem 142. *If G contains a subdivision of H, say H', then H also is a minor of G.*

Theorem 143. *If H has maximum degree at most 3, then H is a minor of G if and only if G contains a subdivision of H.*

Theorem 144 (Wagner 1937)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Theorem 145. *Every 4-connected planar graph is hamiltonian-connected.*

Theorem 146. *The crossing number of K_6 is $\nu(K_6) = 3$.*

Theorem 147. *The crossing number of* $K_{2,2,3}$ *is* $\nu(K_{2,2,3}) = 2$.

Theorem 148. *Let* G *be a graph with* n *vertices and* m *edges. If* k *is the maximum number of edges in a planar subgraph of* G, *then* $\nu(G) \geq m - k$. *Furthermore,* $\nu(G) \geq \frac{m^2}{2k} - \frac{m}{2}$.

Theorem 149. *For* $1 \leq n \leq 10$,

$$\nu(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor.$$

Theorem 150 (R. Guy 1972)**.**

$$\frac{1}{80}n^4 + O(n^3) \leq \nu(K_n) \leq \frac{1}{64}n^4 + O(n^3).$$

Theorem 151. *If* s *and* t *are integers* $(s \leq t)$ *and either* $s \leq 6$ *or* $s = 7$ *and* $t \leq 10$, *then*

$$\nu(K_{s,t}) = \lfloor\frac{s}{2}\rfloor\lfloor\frac{s-1}{2}\rfloor\lfloor\frac{t}{2}\rfloor\lfloor\frac{t-1}{2}\rfloor.$$

Theorem 152. *For all* $t \geq 3$,

$$\nu(C_3 \times C_t) = t.$$

Theorem 153. *For all* $t \geq 4$,

$$\nu(C_4 \times C_t) = 2t.$$

Theorem 154. *For all* $t \geq 3$,

$$\nu(K_4 \times C_t) = 3t.$$

Theorem 155 (Ajtal–Chvátal–Newborn–Szemerédi 1982, Leighton 1983)**.** *Let G be a simple graph with n verticrs and m edges. If $m \geq 4n$, then*

$$\nu(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Theorem 156 (Spencer–Szemerédi–Trotter 1984)**.** *There are at most $4n^{4/3}$ pairs of points at distance 1 among a set of n points in the plane.*

Theorem 157. *A simple graph G with n vertices and m edges has thickness at least $\frac{m}{3n-6}$. If G has no triangle, then it has thickness at least $\frac{m}{2n-4}$.*

Theorem 158. *The thickness of Q_n is given by*

$$\theta(Q_n) = \lceil \frac{n+1}{4} \rceil.$$

Theorem 159. *For every nonnegative integer k, there exists a connected graph that has a 2-cell embedding on S_k.*

Theorem 160. *Let G be a connected pseudograph of order n and size m with a 2-cell embedding on the surface of genus g and having r regions. Then $n - m + r = 2 - 2g$.*

Theorem 161. *Let G be a connected graph of order n and size m with a 2-cell embedding on the surface of genus g and having r regions. Then $n - m + r = 2 - 2g$.*

Theorem 162. *Every simple graph of order n embedded on S_g has at most $3(n - 2 + g)$ edges.*

Theorem 163 (Heawood's Formula, Heawood 1890). *If G is embeddable on S_g with $g > 0$, then*

$$\chi(G) \leq \lfloor \frac{7 + \sqrt{1 + 48g}}{2} \rfloor.$$

Theorem 164 (The Graph Minor Theorem, Robertson–Seymour 1985). *In any infinite list of graphs, some graph is a minor of another.*

Theorem 165. *For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|G|}{\alpha(G)}$.*

Theorem 166 (Vizing 1963, Aberth 1964). $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$.

Theorem 167 (Welsh–Powell 1967). *If a graph G has degree sequence $d_1 \geq \cdots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}$.*

Theorem 168. *If G is an interval graph, then $\chi(G) = \omega(G)$.*

Theorem 169 (Greedy Coloring). *For every graph G,*

$$\chi(G) \leq 1 + \Delta(G).$$

Theorem 170 (Brooks 1941). *If G is a connected graph that is neither an odd cycle nor a complete graph, then*

$$\chi(G) \leq \Delta(G).$$

Theorem 171. *If G is a k-critical graph, then $\delta(G) \geq k - 1$.*

Theorem 172 (Szekeres–Wilf 1968)**.** *For every graph G,*

$$\chi(G) \leq 1 + \max \delta(G),$$

where the maximum is taken over all induced subgraphs H of G.

Theorem 173 (Gallai 1968, Roy 1967, Vitaver 1962)**.** *If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of G.*

Theorem 174 (Mycielski 1955)**.** *For every positive integer k, there exists a k-chromatic triangle-free graph.*

Theorem 175. *For every two integers $k \geq 2$ and $l \geq 3$ there exists a k-chromatic graph whose girth exceeds l.*

Theorem 176. *Every k-connected graph with n vertices has at least $\binom{k}{2}$ edges. Equality holds for a complete graph plus isolated vertices.*

Theorem 177. *Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph $T_{n,r}$ is the unique graph with the most edges.*

Theorem 178 (Turán 1941)**.** *Among the simple graphs of order n with no $r+1$-clique, the Turán graph $T_{n,r}$ has the maximum number of edges.*

Theorem 179 (Dirac 1953)**.** *Let G be a graph with $\chi(G) > k$, and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges.*

Theorem 180 (Dirac 1953)**.** *Every k-critical graph is $(k-1)$-edge-connected.*

Theorem 181. *Every k-degenerate graph is $k+1$-colorable.*

Theorem 182. *Every outerplanar graph is 3-colorable.*

Theorem 183 (Art Gallary Theorem, Chvátal 1975, Fisk 1978)**.** *If an art gallary is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible to some guard.*

Theorem 184. *Every planar graph with at most 32 edges is 4-colorable.*

Theorem 185. *If T is a tree with n vertices, then $\chi(T;k) = k(k-1)^{n-1}$.*

Theorem 186 (Chromatic recurrence)**.** *If G is a simple graph and $e \in E(G)$, then*

$$\chi(G;k) = \chi(G-e;k) - \chi(G \cdot e;k).$$

Theorem 187. *If G is a chordal graph, then $\chi(G) = \omega(G)$.*

Theorem 188 (Berge 1960)**.** *Every chordal graph is perfect.*

Theorem 189 (Berge 1960)**.** *Comparability graphs are perfect.*

Theorem 190 (The Perfect Graph Theorem (PGT), Lovász 1972)**.** *A graph G is perfect if and only if $\overline{G}$ is perfect.*

Theorem 191 (Replacement Lemma, Lovász 1972)**.** *Every composition of perfect graphs is perfect.*

Theorem 192 (Lovász 1972)**.** *A graph is perfect if and only if* $\omega(G[A])\alpha(G[A]) \geq |A|$ *for all* $A \subseteq V(G)$*.*

Theorem 193 (König 1916)**.** *If* G *is bipartite, then* $\chi'(G) = \Delta(G)$*.*

Theorem 194 (Vizing 1964, 1965, Gupta 1966)**.** *If* G *is a nonempty graph, then* $\chi'(G) \leq 1 + \Delta(G)$*.*

Theorem 195 (Shannon 1949)**.** *If* G *is a graph, then* $\chi'(G) \leq \frac{3}{2}\Delta(G)$*.*

Theorem 196 (Goldberg 1973, 1984, Seymour 1979)**.** *If* $\chi'(G) \geq \Delta(G) + 2$*, then* $\chi'(G) = \max_{H \subseteq G} \lceil \frac{e(H)}{\lfloor n(H)/2 \rfloor} \rceil$*.*

Theorem 197. *If* G *is a nonempty graph in which the vertices of maximum degree are independent, then* $\chi'(G) \leq \Delta(G)$*.*

Theorem 198. *Every overfull graph is of class two.*

Theorem 199. *Let* G *be a connected graph of class two that is minimal with respect to edge chromatic number. Then every vertex of* G *is adjacent to at least two vertices of degree* $\Delta(G)$*. In particular,* G *contains at least three vertices of degree* $\Delta(G)$*.*

Theorem 200. *Let* G *be a connected graph of class two that is minimal with respect to edge chromatic number. If* u *and* v *are adjacent vertices with* $d(u) = k$*, then* v *is adjacent to at least* $\Delta(G) - k + 1$ *vertices of degree* $\Delta(G)$*.*

Theorem 201. *If* G *is a planar graph with* $\Delta(G) \geq 10$*, then* G *is of class one.*

Theorem 202. *If G is a nonempty graph of order n and t is an integer for which $t! > n$, then*

$$\chi''(G) \leq \chi'(G) + t + 2.$$

Theorem 203. *If G is a nonempty graph of order n and t is an integer satisfying $t! > n$, then*

$$\chi''(G) \leq \Delta(G) + t + 3.$$

Theorem 204. *Every planar graph is 4-colorable if and only if every plane graph is 4-face-colorable.*

Theorem 205 (Five Color Theorem, Heawood 1890)**.** *Every planar graph is 5-colorable.*

Theorem 206 (Four Color Theorem, Appel–Haken–Koch 1977)**.** *Every planar graph is 4-colorable.*

Theorem 207. *Every planar graph is 4-colorable if and only if K_5 is a subcontraction of every 5-chromatic graph.*

Theorem 208. *Every planar graph is 4-colorable if and only if every bridgeless cubic planar graph is 3-edge-colorable.*

Theorem 209. *Every bridgeless cubic planar graph is 3-edge colorable.*

Theorem 210. *Every planar graph is 4-colorable if and only if every bridgeless planar graph has a nowhere-zero 4-flow.*

Theorem 211. *Every bridgeless planar graph has a nowhere-zero* 4*-flow.*

Theorem 212. *The Petersen graph has no nowhere-zero* 4*-flow.*

Theorem 213. *Let G be a bridgeless cubic graph. Then G has a nowhere-zero* 4*-flow if and only if G is* 3*-edge-colorable.*

Theorem 214. *Every component of the symmetric difference of two matchings is a path or an even cycle.*

Theorem 215 (Berge 1957)**.** *A matching M in a graph G is a maximum matching if and only if there exists no M-augmenting path in G.*

Theorem 216 (P. Hall 1935, Set System Version)**.** *A collection $S_1, S_2, ..., S_k$, $k \geq 1$, of finite nonempty sets has a system of distinct representatives if and only if the union of any j of these sets contains at least j elements, for each j such that $1 \leq j \leq k$.*

Theorem 217 (P. Hall 1935, Bipartite Graph Version)**.** *An X, Y-bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Theorem 218. *For $k > 0$, every k-regular bipartite graph has a perfect matching.*

Theorem 219 (Ford–Fulkerson 1958)**.** *Families $A = \{A_1, ..., A_m\}$ and $B = \{B_1, ..., B_m\}$ have a common system of distinct representatives (CSDR) if and only if*

$$|(\bigcup_{i \in I} A_i) \bigcap (\bigcup_{j \in J} B_j)| \geq |I| + |J| - m$$

for each pair $I, J \subseteq [m]$.

Theorem 220 (Max-flow Min-cut Theorem, Ford–Fulkerson 1956)**.** *In every network, the maximum value of a feasible flow equals the minimum capacity of a source/sink cut.*

Theorem 221 (Integrality Theorem)**.** *If all capacities in a network are integer, then there is a maximum flow assigning integral flow to each edge. Furthermore, some maximum flow can be partitioned into flows of unit value along paths from source to sink*

Theorem 222 (Tutte's 1-Factor Theorem, Tutte 1947)**.** *A nontrivial graph* G *has a* 1*-factor if and only if for every proper subset* S *of* $V(G)$*, the number of odd components of* $G - S$ *does not exceed* $|S|$.

Theorem 223 (Petersen 1891)**.** *Every bridgeless cubic graph contains a* 1*-factor.*

Theorem 224. *If all the bridges of a connected cubic graph* G *lie on a single path of* G*, then* G *has a* 1*-factor.*

Theorem 225 (Petersen 1891)**.** *Every regular graph with positive even degree has a* 2*-factor.*

Theorem 226. *Let* G *be a* k*-regular graph of even order that remains connected when any* $k-2$ *edges are deleted. Then* G *has a* 1*-factor.*

Some notations used in this course only for Theorem 227 $\sim$ 230 :

$\alpha(G)$: vertex covering number

$\beta(G)$: vertex independence number

$\alpha_1(G)$: edge covering number

$\beta_1(G)$: edge independence number

Theorem 227. *In a graph G, $S \subseteq V(G)$ is an independent set if and only if $\overline{S}$ is a vertex cover, and hence $\alpha(G) + \beta(G) = |G|$.*

Theorem 228 (Gallai 1959)**.** *If G is a graph without isolated vertices, then*

$$\alpha_1(G) + \beta_1(G) = |G|.$$

Theorem 229 (König–Egerváry 1931)**.** *If G is a bipartite graph, then the maximum size $\beta_1(G)$ of a matching in G is equal to the minimum size $\alpha(G)$ of a vertex cover of G.*

Theorem 230 (König 1916)**.** *If G is a bipartite graph with no isolated vertices, then $\beta(G) = \alpha_1(G)$.*

Theorem 231. *Every regular bipartite graph of degree $r \geq 1$ is 1-factorable.*

Theorem 232. *The complete graph K_{2k} is 1-factorable.*

Theorem 233. *A graph G is 2-factorable if and only if G is $2k$-regular for some integer $k \geq 1$.*

Theorem 234. *For every positive integer k, the graph K_{2k+1} is hamiltonian factorable.*

Theorem 235. *The complete graph K_{2k} can be factored into k hamiltonian paths.*

Theorem 236. *Let r and n be integers with $0 \leq r \leq n-1$. Then there exists an r-regular graph of order n if and only if r and n are not both odd.*

Theorem 237. *The graph K_{2k} can be factored into $k-1$ hamiltonian cycles and a 1-factor.*

Theorem 238. *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

Theorem 239. *A nontrivial connected graph is $2K_2$-decomposable if and only if G has even size m, $\Delta(G) \leq \frac{1}{2}m$, and $G \neq K_3 \bigcup K_2$.*

Theorem 240 (Kirkman)**.** *The complete graph K_n is K_3-decomposable if and only if n is odd and $3|\binom{n}{2}$.*

Theorem 241. *For every graph H without isolated vertices, there exists a regular H-decomposable graph.*

Theorem 242. *If H is a graceful graph of size m, then K_{2m+1} is H-decomposable. Indeed, K_{2m+1} can be cyclically decomposed into copies of H.*

Theorem 243. *If F is a linear forest of size k hoving no isolated vertices, then K_{2k} is F-decomposable.*

Theorem 244. *If G is a graceful eulerian graph of size m, then $m \equiv 0 \,(mod\, 4)$ or $m \equiv 3 \,(mod\, 4)$.*

Theorem 245. *The cycle C_n is graceful if and only if $n \equiv 0 \,(mod\, 4)$ or $n \equiv 3 \,(mod\, 4)$.*

Theorem 246. *The complete graph K_n $(n \geq 2)$ is graceful if and only if $n \leq 4$.*

Theorem 247. *Every complete bipartite graph is graceful.*

Theorem 248. *Every nontrivial path is graceful.*

Theorem 249. *If G is a graph without isolated vertices and S is a minimal dominating set of G, then $V(G) - S$ is a dominating set of G.*

Theorem 250. *If G is a graph of order n without isolated vertices, then $\gamma(G) \leq n/2$.*

Theorem 251 (Arnautov 1974, Payan 1975)**.** *Let G be a graph of order n with $\delta = \delta(G) \geq 2$. Then*

$$\gamma(G) \leq \frac{n(1 + \ln(\delta + 1))}{\delta + 1}.$$

Theorem 252. *Every graph G without isolated vertices contains a minimum dominating set S such that for every vertex v of S, there exists a vertex w of $G - S$ such that $N(w) \bigcap S = v$.*

Theorem 253. *If G is a graph of order n, then*

$$\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n - \Delta(G).$$

Theorem 254. *If G is a graph of order n, then*

$$\gamma(G) \leq n - \kappa(G).$$

Theorem 255. *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \min\{\alpha(G), \alpha_1(G), \beta(G), \beta_1(G)\}.$$

Theorem 256. *If K_n is factored into G_1, G_2 and G_3, then*

$$\gamma(G_1) + \gamma(G_2) + \gamma(G_3) \leq 2n + 1.$$

Theorem 257. *If G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ has isolated vertices, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \frac{n+4}{2}.$$

Theorem 258. *Every connected 3-domination maximal graph of even order has a 1-factor.*

Theorem 259. *Every connected 3-domination maximal graph of order at least 7 contains a hamiltonian path.*

Theorem 260. *A set S of vertices in a graph is an independent dominating set if and only if S is maximal independent.*

Theorem 261. *Every maximal independent set of vertices in a graph is a minimal dominating set.*

Theorem 262 (Mantel 1907)**.** *Every graph of order $n \geq 3$ and size at least $\lfloor n^2/4 \rfloor + 1$ contains a triangle.*

Theorem 263. *Let r and n be positive integers, where $n \geq r \geq 2$. Then every graph of order n and size at least*

$$(\frac{r-2}{2r-2})n^2 + 1$$

contains K_r as a subgraph.

Theorem 264 (Dirac 1952)**.** *Every graph of order $n \geq 4$ and size at least $2n - 2$ contains a subdivision of K_4 as a subgraph.*

Theorem 265. *Every graph of order $n \geq 6$ and size at least $3n - 5$ contains two disjoint cycles.*

Theorem 266. *Every graph of order $n \geq 5$ and size at least $n + 4$ contains two edge-disjoint cycles.*

Theorem 267. *If G is a graph of order n and size m with $n \geq 4$ and*

$$m \geq \frac{n + n\sqrt{4n-3}}{4} + 1,$$

then G contains a 4-cycle.

Theorem 268. *If G is a graph of order n and size m with $n \geq 4$ and*

$$m \geq \frac{n + \sqrt{n-1}}{2} + 1,$$

then G contains a 3-cycle or a 4-cycle.

Theorem 269. *Let k and n be integers with $1 \leq k < n$. Every graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1$$

contains a subgraph with minimum degree k.

Theorem 270. *Let k and n be integers with $1 \leq k < n$. If G is a graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1,$$

then G contains every tree of size k as a subgraph.

Theorem 271. *For positive integers n and k with $n \geq 2k$,*

$$ex(n; kK_2) = \max\{(k-1)n - \binom{k}{2}, \binom{2k-1}{2}\}.$$

Theorem 272. *Let n and k be positive integers with $n \geq 2k$. Every graph of order n and size at least*

$$\max\{(k-1)n - \binom{k}{2}, \binom{2k-1}{2}\}$$

contains every forest of size k without isolated vertices as a subgraph.

Theorem 273. *For $r \geq 2$, $f(r,4) = 2r$. Furthermore, there is only one $(r,4)$-cage; namely, $K_{r,r}$.*

Theorem 274. *The Petersen graph is the unique 5-cage.*

Theorem 275. *If G is an r-regular Moore graph ($r \geq 3$) of girth 5, then $r = 3$, $r = 7$ or, possibly, $r = 57$.*

Theorem 276. *The Ramsey number $R(3,3) = 6$.*

Theorem 277. *For every two positive integer s sand t, the Ramsey number $R(s,t)$ exists; moreover,*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Theorem 278. *For integers $s \geq 2$ and $t \geq 2$,*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

If both summands on the right are even, then the inequality is strict.

Theorem 279. *For every integer $t \geq 3$,*

$$R(3,t) \leq \frac{t^2+3}{2}.$$

Theorem 280 (Erdős 1947)**.**

$$R(p,p) > (e\sqrt{2})^{-1} p 2^{p/2}(1 + o(1)).$$

Theorem 281 (Erdős 1947)**.** *If* $\binom{n}{p}2^{1-\binom{p}{2}} < 1$*, then*

$$R(p,p) > n.$$

Theorem 282. *For every integer* $t \geq 3$*,*

$$R(t,t) \geq \lfloor 2^{t/2} \rfloor.$$

Theorem 283. *Let the graphs* $G_1, G_2, ..., G_k$ $(k \geq 2)$ *be given. Then the Ramsey number* $R(G_1, G_2, ..., G_k)$ *exists.*

Theorem 284 (Chvátal 1977)**.** *Let* T_s *be any tree of order* $s \geq 1$ *and let* t *be a positive integer. Then*

$$R(T_s, K_t) = 1 + (s-1)(t-1).$$

Theorem 285 (Burr–Erdős–Spencer 1975)**.** *For* $m \geq 3$*,*

$$R(mK_3, mK_3) = 5m.$$

Theorem 286. *Let* $s_1, s_2, ..., s_k (k \geq 2)$ *be positive integers,* t *of which are even. Then*

$$R(K_{1,s_1}, K_{1,s_2}, ..., K_{1,s_k}) = \sum_{i=1}^{k}(s_i - 1) + \theta_t,$$

where $\theta_t = 1$ *if* t *is positive and even and* $\theta_t = 2$ *otherwise.*

Theorem 287. *For all positive integers* s *and* t*, if* $G \to (K_s, K_t)$*, then* $\chi(G) \geq \chi(K_r)$*, where* $r = R(s,t)$*.*

Theorem 288. *For all positive integers s and t, if $G \to (K_s, K_t)$, then the size of G is at least $\binom{r}{2}$, where $r = R(s,t)$.*

Theorem 289. *For every two positive integers $s \geq 2$ and $t \geq 2$, the irredundant Ramsey number $IR(s,t)$ exists; moreover,*

$$IR(s,t) \leq IR(s-1,t) + IR(s,t-1).$$

Theorem 290. *The irredundant Ramsey number $IR(3,3) = 6$.*

Theorem 291. *For every two positive integers s and t, the lower Ramsey number $LR(s,t)$ exists; moreover,*

$$s + t + 1 \leq LR(s,t) \leq 2(s+t) - 1.$$

Theorem 292. *For every positive integer k, the lower Ramsey number $LR(1,k^2) = k^2 + 2k$.*

Theorem 293. *For every two positive integers s and t the bipartite Ramsey number $BR(s,t)$ exists; moreover,*

$$BR(s,t) \leq \binom{s+t}{s} - 1.$$

Theorem 294. *For integers $s \geq 2$ and $t \geq 2$,*

$$BR(s,t) \leq BR(s-1,t) + BR(s,t-1) + 1.$$

Theorem 295. *For every positive integer t,*

$$BR(t,t) \leq 2^t(t-1) + 1.$$

Theorem 296. *For every positive integer k and sufficiently large integer n, there is a tournament T of order n with property S_k.*

Theorem 297 (Szele 1943)**.** *For each positive integer n there is a tournament of order n with at least $n!/2^{n-1}$ hamiltonian paths.*

Theorem 298 (Erdős 1959)**.** *For each positive integer n there is a tournament of order n with at least $n!/2^{n-1}$ hamiltonian paths.*

Theorem 299. *For any fixed positive real number $p < 1$, almost all graphs are connected with diameter 2.*

Theorem 300. *For fixed nonnegative integers i and j and a positive real number $p < 1$, almost all graphs have the property that if S and T are disjoint i-element and j-element subsets of vertices, then there is a vertex $z \notin S \bigcup T$ that is adjacent to every vertex of S and to no vertex of T.*

Theorem 301. *For fixed nonnegative integers i and j and a positive real number $p < 1$; almost all graphs have property $P_{i,j}$.*

Theorem 302. *For each graph H and fixed positive real number $p < 1$, almost all graphs contain H as an induced subgraph.*

Theorem 303. *For any fixed positive real number $p < 1$, almost no graphs are planar.*

Theorem 304. *For any positive integer k and fixed positive real number $p < 1$, almost no graphs are k-colorable.*

Theorem 305. *For any positive integer k and fixed positive real number $p < 1$, almost no graphs have genus k.*

Theorem 306 (Gilbert 1959)**.** *For any fixed positive integer k and positive real number $p < 1$, almost all graphs are connected.*

Theorem 307. *For any fixed positive integer k and positive real number $p < 1$, almost all graphs are k-connected.*

Theorem 308. *If two vertices are nonadjacent in the Petersen graph, then they have exactly one common neighbor.*

Theorem 309 (Tait 1878)**.** *A simple 2-edge-connected 3-regular plane graph is 3-edge-colorable if and only if it is 4-face-colorable.*

Theorem 310 (Tutte 1949)**.** *A cubic graph has a nowhere-zero 3-flow if and only if it is bipartite.*

Theorem 311 (Tutte 1945)**.** *A plane bridgeless graph is k-face colorable if and only if it has a nowhere-zero k-flow.*

Theorem 312. *In a minimal imperfect graph, no stable set intersects every maximum clique.*

Theorem 313 (Erdős–Szekeres 1935)**.** *Every list of more than n^2 distinct numbers has a monotone sublist of length more than n.*

Theorem 314 (Graham–Kleitman 1973). *In every labeling of $E(K_n)$ using distinct integers, there is a trail of length at least $n-1$ along which the labels strictly increase.*

Theorem 315 (Ramsey 1930). *Given positive integers r and $p_1, p_2, ..., p_k$, there exists an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i.*

Theorem 316 (Erdős–Szekeres 1935). *Given an integer m, there exists a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane with no three collinear contains an m-subset forming a convex m-gon.*

Theorem 317 (Sperner's Lemma 1928). *Every properly labeled simplicial subdivision has a completely labeled cell.*

Theorem 318 (Erdős–Goodman–Pósa 1966). *The intersection number equals the minimum number of complete subgraphs needed to cover $E(G)$.*

Theorem 319 (McGuinness 1994). *Every greedy clique decomposition of a graph with n vertices uses at most $\lfloor n^2/4 \rfloor$ cliques.*

Theorem 320 (Winkler 1983). *Every connected graph of order n has squashed-cube dimension at most $n-1$.*

Theorem 321 (Erdős–Rubin–Taylor 1979). *If $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not k-choosable.*

Theorem 322 (Thomassen 1994). *Planar graphs are 5-choosable.*

Theorem 323 (Lovász 1968). *Every graph of order n can be decomposed into $\lfloor n/2 \rfloor$ paths and cycles.*

Theorem 324. *The diameter of a connected graph G is less than the number of distinct eigenvalues of G.*

Theorem 325 (Hoffman 1963). *A graph G is regular and connected if and only if J is a linear combination of powers of $A(G)$.*

Theorem 326. $\phi(\overline{G}; \lambda) = (-1)^n det[(-\lambda - 1)I - A(G) + J]$.

Theorem 327. *For a k-regular simple graph, $\lambda_n \geq k - n$.*

Theorem 328. *If G is a strongly regular graph with n vertices and parameters k, λ, μ, then $k(k - \lambda - 1) = \mu(n - k - 1)$.*

Theorem 329. *A k-regular connected graph G is strongly regular with parameters k, λ, μ if and only if it has exactly three eigenvalues $k > r > s$ and these satisfy $r + s = \lambda - \mu$ and $rs = -(k - \mu)$.*

Theorem 330 (Friendship Theorem, Wilf 1971). *If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.*