On Optimal Pebbling of Hypercubes

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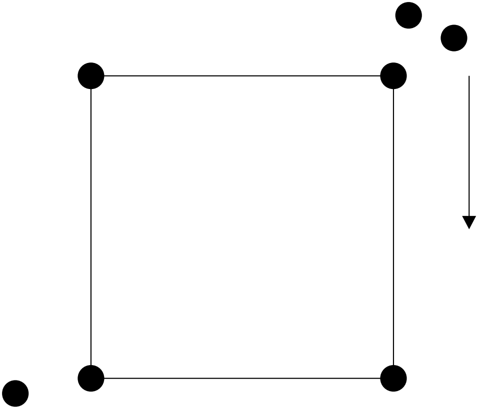
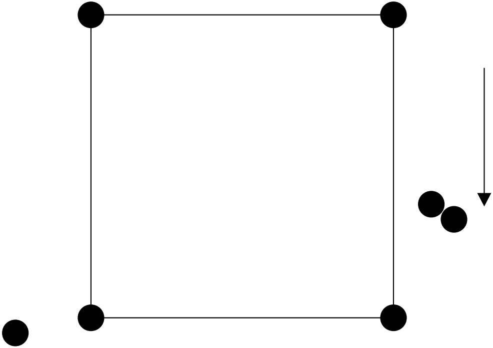
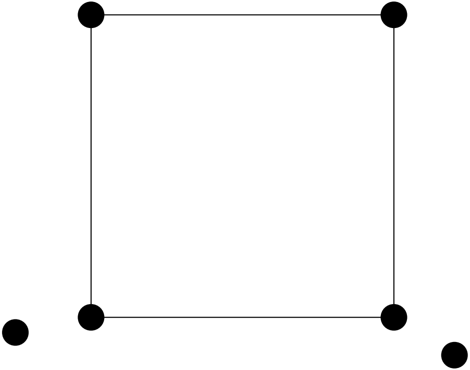
Let G be a graph such that $V(G) = \{v_1, v_2, \ldots, v_p\}$. By a *distribution* of pebbles on G we mean a function $\delta : V(G) \to N \cup \{0\}$ and for clarity, we use $(\delta_{v_1}, \delta_{v_2}, \ldots, \delta_{v_p})$ to denote δ, where δ_v is the number of pebbles distributed on $v \in V(G)$. The *support* S_δ of δ is defined as the set of vertices v in $V(G)$ such that $\delta_v > 0$. Therefore the number of pebbles used in G is $\sum_{v \in S_\delta} \delta_v$ and denoted by δ_G.

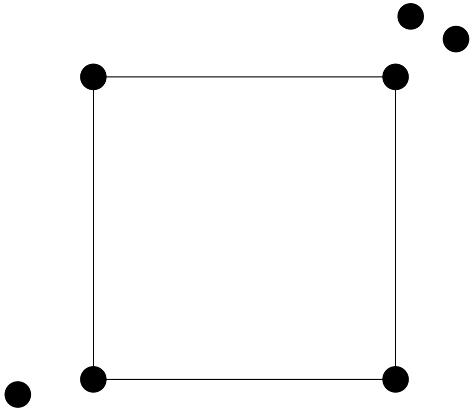
A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution δ of pebbles lets us move at least one pebble to each vertex v by applying pebbling moves repeatedly(if necessary), then δ is called a pebbling of G. The *optimal pebbling number* $f'(G)$ of G is the minimum number of pebbles used in a pebbling of G.







A pebbling of C_4.



The *pebbling number* $f(G)$ of G is the minimum number of pebbles k such that **any distribution** of k pebbles is a pebbling of G.

$$f(G) \geq \max\{V(G), 2^{diam(G)}\}.$$

The problem of pebbling graph was first proposed by J. Lagarias and M. Saks as a tool for solving a number theoretic problem by Lemke and Kleitman [5]. Since then, quite a few of work has been done by F. R. K. Chung [1], Guzman, Moews [7], Pachter [9], Clarke et. al. [2] and Herscovici et. al. [4].

References

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The optimal pebbling number of a graph G was first introduced by Pachter and the following results are noteable.

Theorem

Let P be a path with $3t + r$ vertices with $0 \leq r \leq 2$. Then $f'(P) = 2t + r$.

Theorem (Fu and Shiue)

For any two graphs G and H, $f'(G \times H) \le f'(G)f'(H)$.

Theorem (Fu and Shiue, DM)

Let T_h^m be a complete m-ary tree with height h. Then $f'(T_h^m) = 2^h$ for each $m \ge 3$, and

$$f'(T_h^2) = \min\{\sum_{i=0}^{h} 2^i x_i | \sum_{i=0}^{h} (2^i - \frac{1}{3})x_i \ge \frac{1}{3} \cdot 2^{h+1}, x_0 \in \{0,1,2,3\} \text{ and } x_i \in \{0,2\}, \text{where } i = 1,2,\ldots,h\}.$$

Theorem (Moews, DM)

Let Q_n be the hypercube defined by $Q_n = Q_{n-1} \times K_2$. Then $f'(Q_n) = (\frac{4}{3})^{n+O(\log n)}$.

In fact, the upper bound of $f'(Q_n)$ obtained by Moews is as follows.

Corollary

$f'(Q_n) \leq 2(\frac{4}{3})^n n^2$.

Let the *covering radius* of a subset W of $V(G)$ be the smallest positive integer d such that all vertices v of G are at distance no more than d from a vertex of W. Then, it is clear that we can place 2^d pebbles on each vertex of W and obtain a pebbling of G with $2^d|W|$ pebbles. Therefore, the choice of W determines an upper bound for $f'(Q_n)$.

Suppose $0 < \beta < \frac{1}{2}$. Let $d(v, W)$ be the minimum values of $d(v, w)$ for $w \in W$ and p be the probability for a vertex $v \in V(Q_n)$ satisfying that $d(v, W) > \beta n$. Then, we have

Lemma

$$p \leq \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k .$$

Proof.

Set $\mathcal{U}_k = \{S : S \text{ is a } k\text{-element subset of } V(Q_n)\}$ and for each $v \in V(Q_n)$ $S_v = \{u \in V(Q_n) : d(u,v) > \beta n\}$. Then

$$|S_v| = \sum_{i=\beta n+1}^{n} \binom{n}{i} = 2^n - \sum_{i=0}^{\beta n} \binom{n}{i}.$$

It is easy to see that $d(v, W) > \beta n$ for $W \in \mathcal{U}_k$ if and only if $W \subseteq S_v$. Let $\mathcal{F}_v = \{W : W \text{ is a } k\text{-element subset of } S_v\}$, $q = |V(Q_n)| = 2^n$ and $r = |S_v|$.

Then

$$
\begin{aligned}
p &= \frac{|\mathcal{F}_v|}{|\mathcal{U}_k|} = \frac{\binom{r}{k}}{\binom{q}{k}} \\
&= \frac{r(r-1)(r-2)\cdots(r-k+1)}{q(q-1)(q-2)\cdots(q-k+1)} \\
&\leq \left(\frac{r}{q}\right)^k \\
&= \left[1 - 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}\right]^k.
\end{aligned}
$$

□

Theorem

$f'(Q_n) = O(n^{\frac{3}{2}}(\frac{4}{3})^n)$.

Proof.

Suppose $0 < \beta < \frac{1}{2}$. Let W be a randomly chosen k-element subset of $V(Q_n)$. Let p be the probability for a randomly and uniformly chosen $v \in V(Q_n)$ satisfying that $d(v, W) > \beta n$. Set X the number of vertices in $V(Q_n)$ at distance greater than βn from W. If the probability $Pr[X = 0] > 0$, then there exists a k-element subset of $V(Q_n)$ with covering radius at most βn.

It is clear that the expectation $E[X] = 2^n p$. By Markov's inequality, $Pr[X \geq 1] \leq E[X] = 2^n p$. Hence, $Pr[X = 0] = 1 - Pr[X \geq 1] \geq 1 - 2^n p$. Moreover, by Lemma 6, we have

$$p \leq \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k.$$

So,

$$Pr[X = 0] \geq 1 - 2^n p \geq 1 - 2^n \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k.$$

By directed counting,

$$1 - 2^n \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right]^k > 0$$

if and only if

$$-\log \left[1 - 2^{-n} \sum_{i=0}^{\beta n} \binom{n}{i}\right] > \frac{n \log 2}{k}.$$

From elementary calculus, we have

$$
\begin{aligned}
-\log\left[1-2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}\right] &= 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}+\frac{1}{2}\left[2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}\right]^2+\cdots \\
&\geq 2^{-n}\sum_{i=0}^{\beta n}\binom{n}{i}.
\end{aligned}
$$

Combining with the estimation in [6],

$$\sum_{i=0}^{\beta n} \binom{n}{i} \geq 2^{nH(\beta)}[8n\beta(1-\beta)]^{-\frac{1}{2}} \geq 2^{nH(\beta)}(2n)^{-\frac{1}{2}},$$

we obtain

$$-\log\left[1 - 2^{-n}\binom{n}{\beta n}\right] \geq 2^{-n[1-H(\beta)]}(2n)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} n^{-\frac{1}{2}} 2^{-n[1-H(\beta)]},$$

where $H(t) = -t\log_2 t - (1-t)\log_2(1-t)$ for $0 \leq t \leq \frac{1}{2}$. Now, by letting

$$k \geq (\sqrt{2}\log 2) n^{\frac{3}{2}} 2^{n[1-H(\beta)]},$$

it is easy to see that

$$\frac{1}{\sqrt{2}} n^{-\frac{1}{2}} 2^{-n[1-H(\beta)]} \geq \frac{n \log 2}{k}$$

and then

$$Pr[X = 0] \geq 1 - 2^n \left[1 - 2^{-n} \binom{n}{\beta n} \right]^k > 0.$$

In this case, $f'(Q_n) \leq k2^{\beta n}$. Since $T(\beta) = 1 + \beta - H(\beta)$, where $0 < \beta < \frac{1}{2}$, attains its minimum at $\beta = \frac{1}{3}$ with $T(\frac{1}{3}) = \log_2(\frac{4}{3})$,

$$f'(Q_n) = O(n^{\frac{3}{2}} 2^{n \log_2(\frac{4}{3})}) = O(n^{\frac{3}{2}} (\frac{4}{3})^n).$$

This concludes the proof. □

A pebbling δ of G is called an *ℓ-pebbling* if we can move at least ℓ pebbles to each vertex of G. The *optimal ℓ-pebbling number* $f'_\ell(G)$ of G is defined as the minimum number of pebbles in an ℓ-pebbling of G.

Theorem (Shiue)

Let P be a path with n vertices and $\ell = 3m + r \geq 2,\ 0 \leq r \leq 2$. Then $f'_\ell(P) = m(n+2) + \lceil \frac{r(n+2)}{2} \rceil$.

Lemma

$f'(Q_{n+m}) \leq f'_{2^{\lceil \frac{n-1}{2} \rceil}}(Q_m) + f'_{2^{\lfloor \frac{n-1}{2} \rfloor}}(Q_m)$.

Theorem

For each $n \geq 4$, $f'(Q_n) \leq \min\{\frac{3}{2}[1 + (\frac{3}{2})^m \cdot 2^{-t}](\frac{9}{8})^t(\frac{4}{3})^n | n = m + 2t + 1\}$, *where* m *and* t *are positive integers.*

Theorem

$f'(Q_n) = O((1.377)^n)$

This is a better result for small n.

Thank you for your attention!