Discrete Math, Test(I)

2009.4.8

1. 令 D_n 代表 n 個元素的亂排(Derangement)數. 求 D_7 .

2. 令|A| = 11, |B| = 7. 試求(a) 由 A 映至 B 的函數個數及(b) 由 A 到 B 的映成函數個數.

3. 令 $\varphi(n)$ 代表小於 n 而且與 n 互質的正整數個數. 證明
   (a) 對於所有 $n \ge 3$, $\varphi(n)$ 爲偶數. (15 分)
   (b) $\sum_{k|n} \varphi(k) = n$. (答對加 10 分)

4. 令 A = {1, 2, …, 100}. 試求在 A 中取出 11 個數使得這些元素都不是連續整數的方法數.

5. 證明
   (a) 把一個正整數分成每一部份都不相等的方法數等於把它分成都是奇數部份的方法數.
   (b) 把一個正整數分成 m 部份的方法數等於把它分成最大部份爲 m 的方法數.

6. 解遞迴關係
   (a) $h_n = 2h_{n-1} + 3^n$, $n \ge 1$, $h_0 = 1$.
   (b) $h_n = h_{n-1} - h_{n-2}$, $h_1 = 1$, $h_2 = 0$.

7. 用 a, b, c 來組成一個長度大不大於 6 的字,其中 b 與 c 最多用 3 次,問分法有多少種.

8. 試求用不相交的對角線把一個凸 12 邊形分成全部都是三角形的方法數.

(只寫答案,答對只給 5 分)

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1. 令 D_n 代表 n 個元素的亂排(Derangement)數. 求 D_7 .
   (Method 1)
   $D_{n+1} = n(D_n + D_{n-1})$ *and* $D_1 = 0, D_2 = 1$
   (Method 2)
   第一列已經排定,第二列部與第一列相同,令 α_i = 第i個位置爲i
   則 $\mathrm{D}_7 = N(\alpha_1{}'\alpha_2{}'\cdots\alpha_7{}') = 7!\,(1-\frac{1}{1}+\frac{1}{2}-\cdots-\frac{1}{7!})$

2. 令|A| = 11, |B| = 7. 試求(a) 由 A 映至 B 的函數個數及(b) 由 A 到 B 的映成函數個數.
   (a) 7^{11}
   (b) $\sum_{i=0}^{6}(-1)^i\binom{7}{i}(7-i)^{11}$

3. 令 $\varphi(n)$ 代表小於 n 而且與 n 互質的正整數個數. 證明
   (a) 對於所有 $n \geq 3$, $\varphi(n)$ 爲偶數. (15 分)
   (b) $\sum_{k|n}\varphi(k) = n$. (答對加 10 分)

$Let\ \ n = p_1^{s_1} p_2^{s_2} \cdots p_e^{s_e}, where\ \ p_i\ is\ a\ prime\ \ and\ \ s_i \geq 1.$

(a) $\varphi(n) = n\prod_{i=1}^{e}(1-\frac{1}{p_i}) = \prod_{i=1}^{e} p_i^{s_i}(p_i - 1)$

$For\ n \geq 3, \exists p_i \geq 3\ and\ \ p_i\ is\ odd\ or\ \ p_i = 2\ and\ \ s_i \geq 2.$

$If\ \ p_i \geq 3\ and\ \ p_i\ \ is\ odd, then\ (p_i - 1)\ is\ even.$

$If\ \ \ p_i = 2\ and\ \ s_i \geq 2, then\ \ p_i^{s_i - 1}\ \ is\ even.$

(b)

$Let\ S = \{1, 2, \cdots, n\}\ and\ \ A(d) = \{k : (k, n) = d,\ 1 \leq k \leq n\}.$

$$If\ d_1 \neq d_2,\ \mathrm{A}(\mathrm{d}_1) \cap \mathrm{A}(\mathrm{d}_2) = \phi,\ \mathrm{S} = \bigcup_{d|n} \mathrm{A}(\mathrm{d}).$$

$$n = \sum_{d|n} |A(d)|\ and\ A(d) = \varphi(\frac{n}{d}),\ then\ n = \sum_{d|n} \varphi(d).$$

4. 令 A = {1, 2, …, 100}. 試求在 A 中取出 11 個數使得這些元素都不是連續整數的方法數.

A={1, 2, …, 100}. Let $S = \{S_1, S_2, ..., S_{11}\}$爲取出的不連續整數,而且 $1 \le S_1 < S_2 < \cdots < S_{11} \le 100$.

因爲$S = \{S_1, S_2, ..., S_{11}\}$不連續,所以$\{S_1, S_2 - 1, ..., S_{11} - 10\}$都不同.

Define $f : S \to B\ where\ B = \binom{100-(11-1)}{11}.\ f\ is\ a\ bijection..$

因此共有$\binom{90}{11}$種選法

5.
(a)
Consider the generating functions.
Let $f(x) = (1+x)(1+x^2)(1+x^3)\cdots$. Then the coefficients of x^n is the number of partitions of n such that all parts are distinct.
Let
$g(x) = (1+x+x^2+x^3+\cdots)(1+x+x^3+x^6+\cdots)(1+x+x^5+x^{10}+\cdots)\cdots = \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots$
Then the coefficients of x^n is the number of partitions of n such that each part are odd.
Furthermore $f(x) = (\frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\cdots) = \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots = g(x)$.
Therefore the number of partitions of both way are the same.

(b)
Let P be the set of all partitions of n with exactly m parts, and Q be the set of all partitions of n with largest part m. Let $p \in P$, i.e.
$P : n = n_1 + n_2 + ... + n_m$, $n_1 \geq n_2 \geq ... \geq n_m$. Consider the Ferrers diagram of P, $F(p)$, an array with ith row has n_i stars. Every row is flush left, i.e. the stars are aligned along the left margin. Then $F^t(p)$ represent a partition of n with largest part m.
Let $q \in Q$. Then we can define $f : P \to Q$ by $f(p) = q$ if $F(p)^t$ represents q.
It's easy to see transpose is an one to one, onto function, thus so is f. Then $|P| = |Q|$.

6.
(a) $h_n = 2h_{n-1} + 3^n, n >= 1, h_0 = 1$
Since it's homo, $x = 2$, $h_n = c_1 2^n$. Let $h_n = \alpha 3^n$ and we have
$\alpha 3^n = 2(\alpha 3^{n-1}) + 3^n$, thus $\alpha = 3$.
Then the general solution of $h_n = c_1 2^n + 3 \cdot 3^n$ can be obtained with $h_0 = 1$,
$h_0 = c_1 + 3 = 1$, thus $c_1 = -2$.
Hence $h_n = -2 \cdot 2^n + 3^{n+1}$.

(b)$h_n = h_{n-1} - h_{n-2}, h_1 = 1, h_2 = 0$
Observe that $h_3 = -1$, $h_4 = -1$, $h_5 = 0$, $h_6 = 1$, $h_7 = 1 = h_1$, $h_8 = 0 = h_2$.
Since h_n is determined only by h_{n-1} and h_{n-2}, we have that

$$h_n = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \pmod 6 \\ -1, & \text{if } n \equiv 3 \text{ or } 4 \pmod 6 \\ 0, & \text{if } n \equiv 2 \text{ or } 5 \pmod 6 \end{cases}$$

7.
Consider the generating function. It is
$(1 + ax + ax^2 + ax^3 + ... + ax^6)(1 + bx + bx^2 + bx^3)(1 + cx + cx^2 + cx^3)$
Since we need to consider all permutations, we use exponential generating functions here:
$(e^x)(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!})^2 = \sum_{n=0}^{\infty} \frac{x^n}{n!}(1 + x + \frac{x^2}{2} + \frac{x^3}{6}) =$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{7}{12}x^4 + \frac{1}{6}x^5 + \frac{1}{36}x^6)$
Then the coefficients of x^n is what we want, for $1 \leq n \leq 6$. They are:

$a_1 = 3$, $a_2 = 9$, $a_3 = 27$, $a_4 = 79$, $a_5 = 221$, $a_6 = 583$.

8.

Any diagonal parts a n-polygon into a k-polygon and a $(n-k+1)$-polygon, thus

$d_n = d_2 d_{n-1} + d_3 d_{n-2} + ... + d_2 d_{n-1} = \sum_{k=2}^{n-1} d_k d_{n-k+1}$

Let $f(x) = \sum_{n=2}^{\infty} d_n x^n = (d_2 x^2 + d_3 x^3 + ...)$. We assume $d_2 = 1$.

Then $f(x)f(x) = (d_2 x^2 + d_3 x^3 + ...)(d_2 x^2 + d_3 x^3 + ...)$

$= d_2 d_2 x^4 + (d_2 d_3 + d_3 d_2) x^5 + ...$

$= d_3 x^4 + d_4 x^5 + ...$

$= x(d_2 x^2 + d_3 x^3 + d_4 x^4 + ...) - d_2 x^3$

$= x f(x) - x^3$

Hence $f(x)^2 - x f(x) + x^3 = 0$, and $f(x) = \frac{x}{2} + \frac{x}{2}\sqrt{1-4x}$.

Note that $(1-4x)^{1/2} = \sum_{i=1}^{1/2} \binom{1/2}{i} (-4)^i x^i$, and the coefficient of x^{n-1} is $(-4)^{n-1}\binom{1/2}{n-1} = -\frac{2}{n-1}\binom{2n-4}{n-2}$.

Thus $d_n = (-\frac{1}{2})(-\frac{2}{n-1})\binom{2n-4}{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$, in particular $d_{12} = 16796$.