Diameters and Wide-Diameters of De Bruijn Graphs

Wide-Diameters of $UB(d, n)$

Diameters of $UG_B(n, m)$ ***for*** $n^2 \leq m \leq n^3$

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Outline

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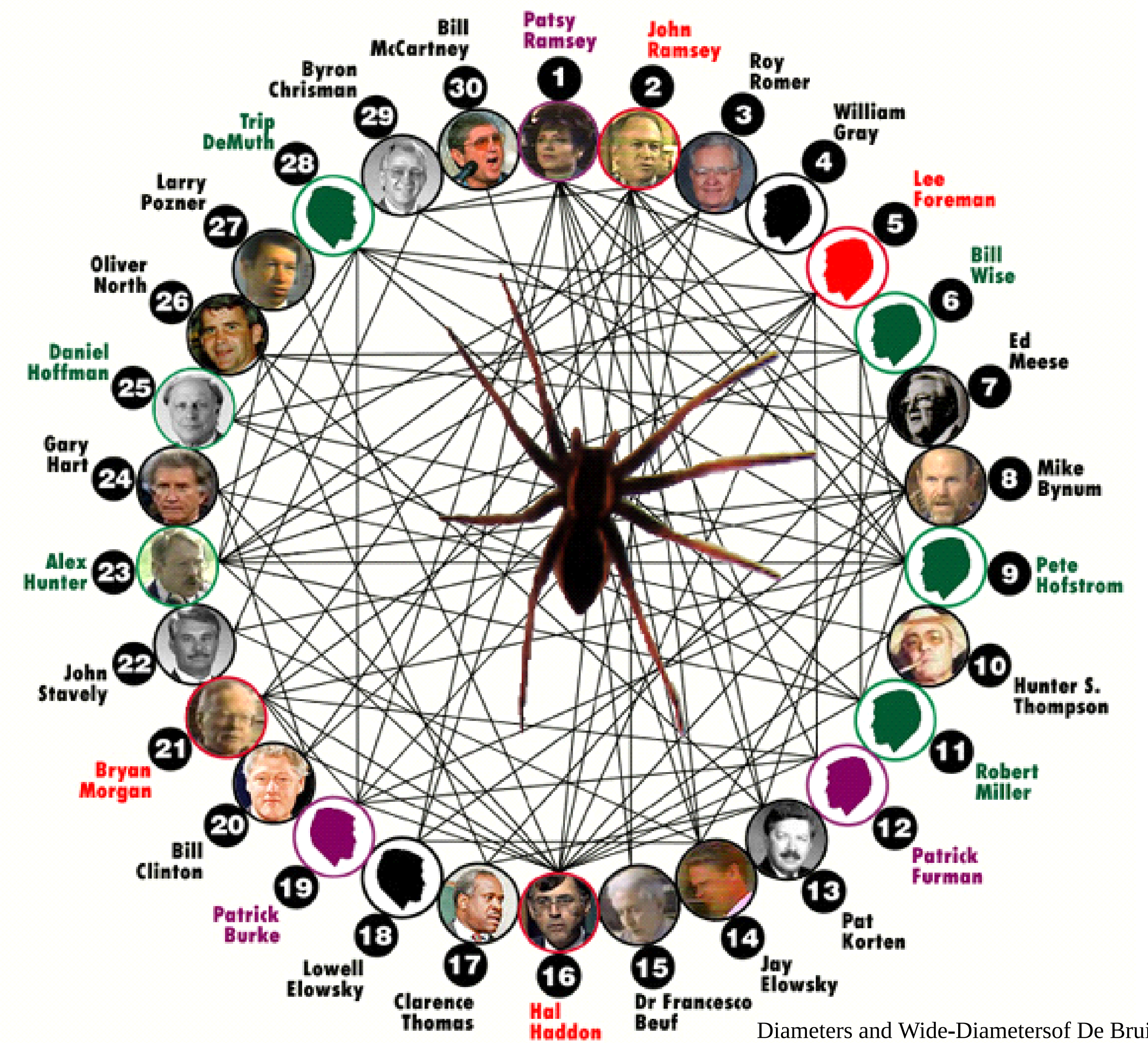
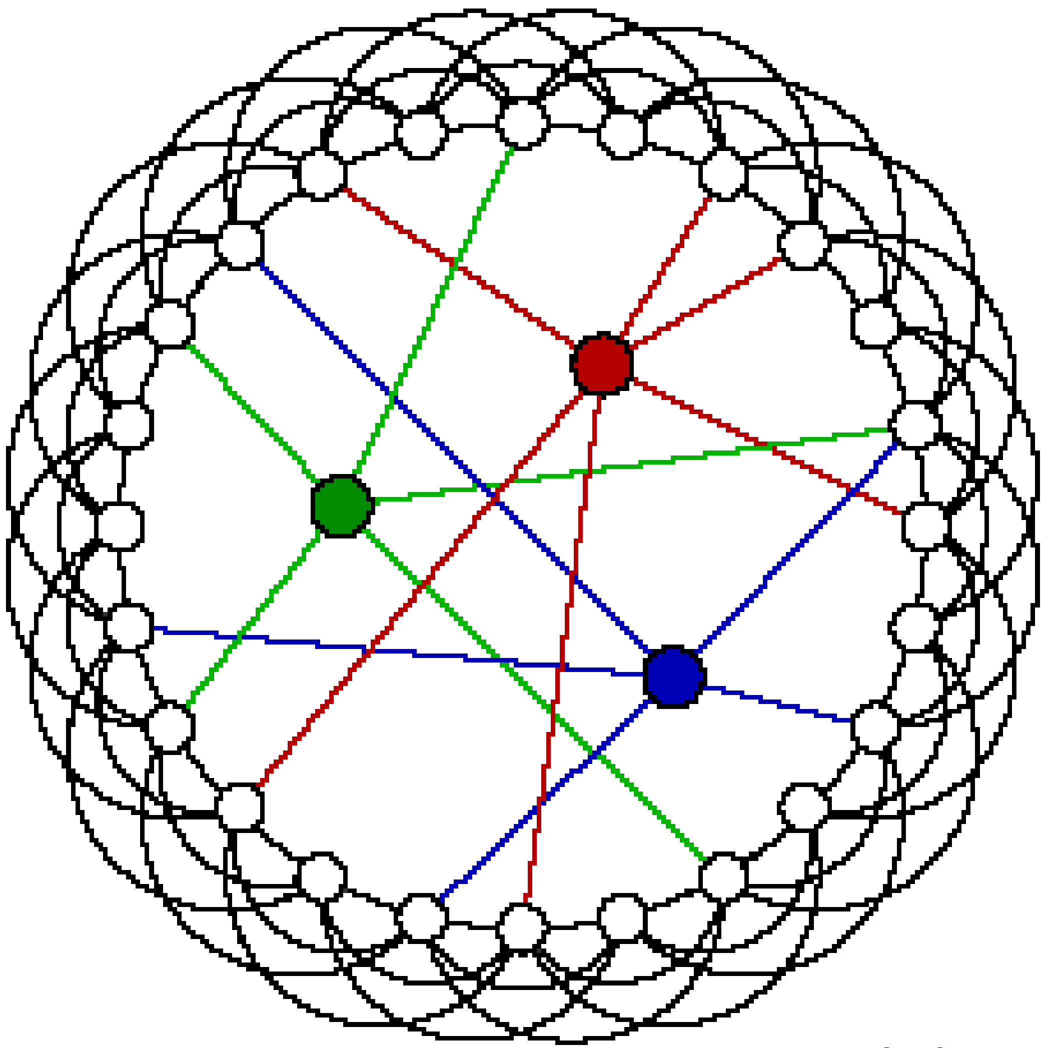
- Known Results I
- The Main Theorem I
- Proof I
- Known Results II
- The Main Results II
- Proof II
- Conclusions
- Future Work

small world, diameter ≤ 6



small diameter

Watts and Strogatz, Nature, 1998



six degrees of separation

directed de Bruijn graphs

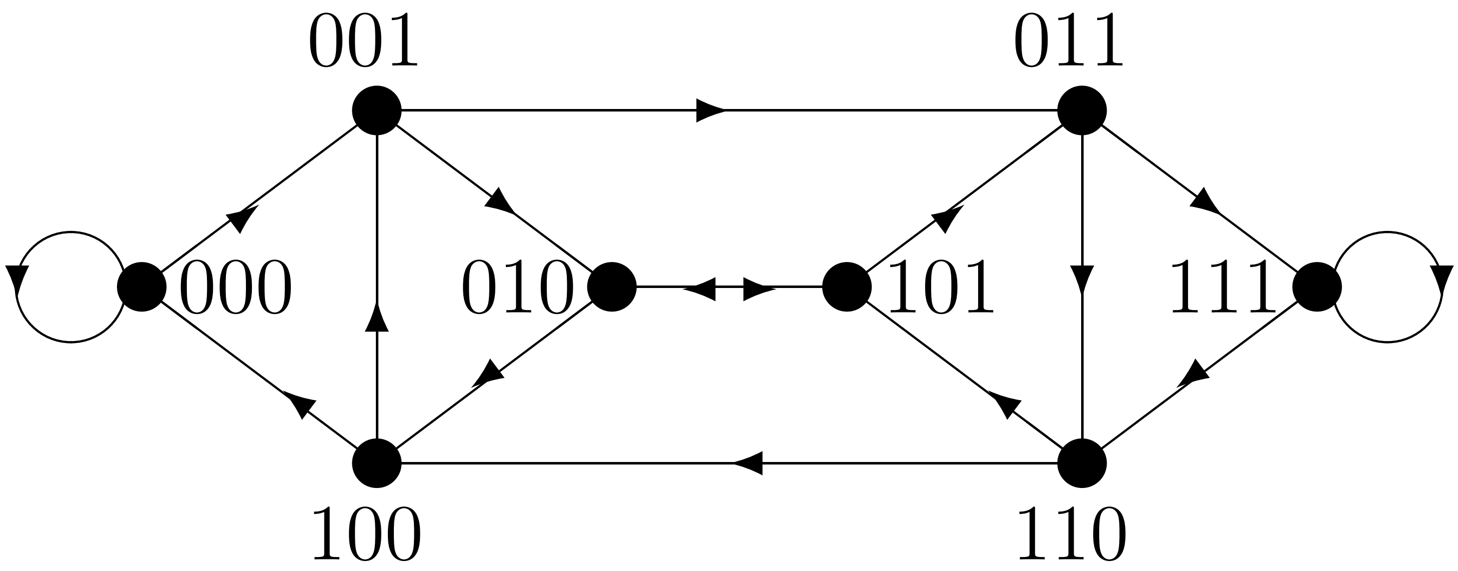


Figure 1: $B(2,3)$.

undirected de Bruijn graphs

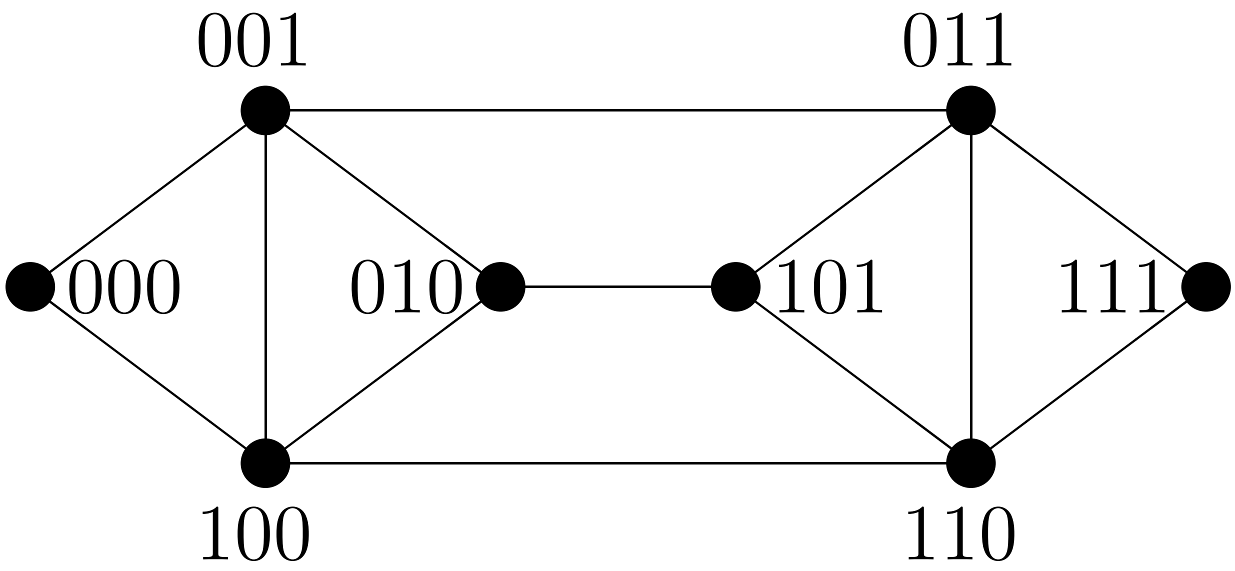


Figure 2: $UB(2,3)$.

Definition $G = B(d, n)$

N.G. de Bruijn 1946

$$V(G) = \{x_1 x_2 \dots x_n : x_i \in Z_d = \{0, 1, \dots, d-1\}\}$$

$$x_1 x_2 \dots x_n \longrightarrow x_2 x_3 \dots x_n y : x_i, y \in Z_d$$

$$Z_d^* = \{1, \dots, d-1\}$$

Definition $G = UB(d, n)$

$$V(G) = \{x_1 x_2 \ldots x_n : x_i \in Z_d = \{0, 1, \ldots, d-1\}\}$$

$$x_1 x_2 \ldots x_n - x_2 x_3 \ldots x_n y$$

$$x_1 x_2 \ldots x_n - z x_1 x_2 \ldots x_{n-1}$$

$$x_i, y, z \in Z_d$$

Diameter of $UB(d,n)$ is n

diameter of $UB(d,n) \leq n$

$$x_1 x_2 \cdots x_n \longrightarrow y_1 y_2 \cdots y_n$$

diameter of $B(d,n) \geq n$

$$000\ldots00 \rightarrow 111\ldots11$$

advantages of $UB(d,n)$

- good connectivity $2d-2$, fault-tolerant routing up to $2d-3$. Jyh-Wen Mao
- short diameter n
- embedding in linear arrays, rings, and complete binary trees
- emulating shuffle-exchanges or hypercubes
- simple routing strategy

disadvantages of $UB(d,n)$

big gap of the number of vertices d^n, d^{n+1}

For example:

$$10^4 \le m \le 10^5$$

$$10^9 \le m \le 10^{10}$$

Definition $G_B(n, m)$

$$i \to in + \alpha \pmod m$$

$$\forall i \in [0, m-1] \text{ and } \forall \alpha \in [0, n-1]$$

$$[a, b] = \{a, a+1, a+2, \ldots, b\}$$

$$V(G) = [0, m-1]$$

Imase and Itoh, 1981
Reddy, Pradhadn and Kuhl, 1980

Figure $G_B(2,7)$

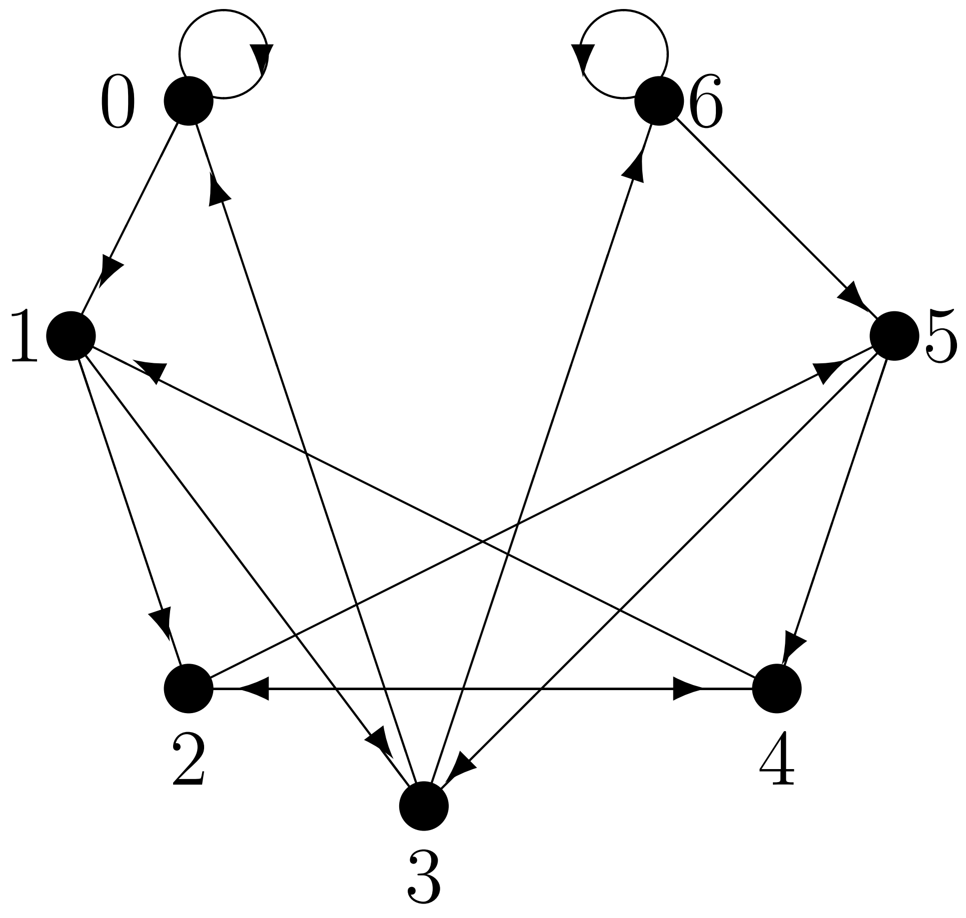
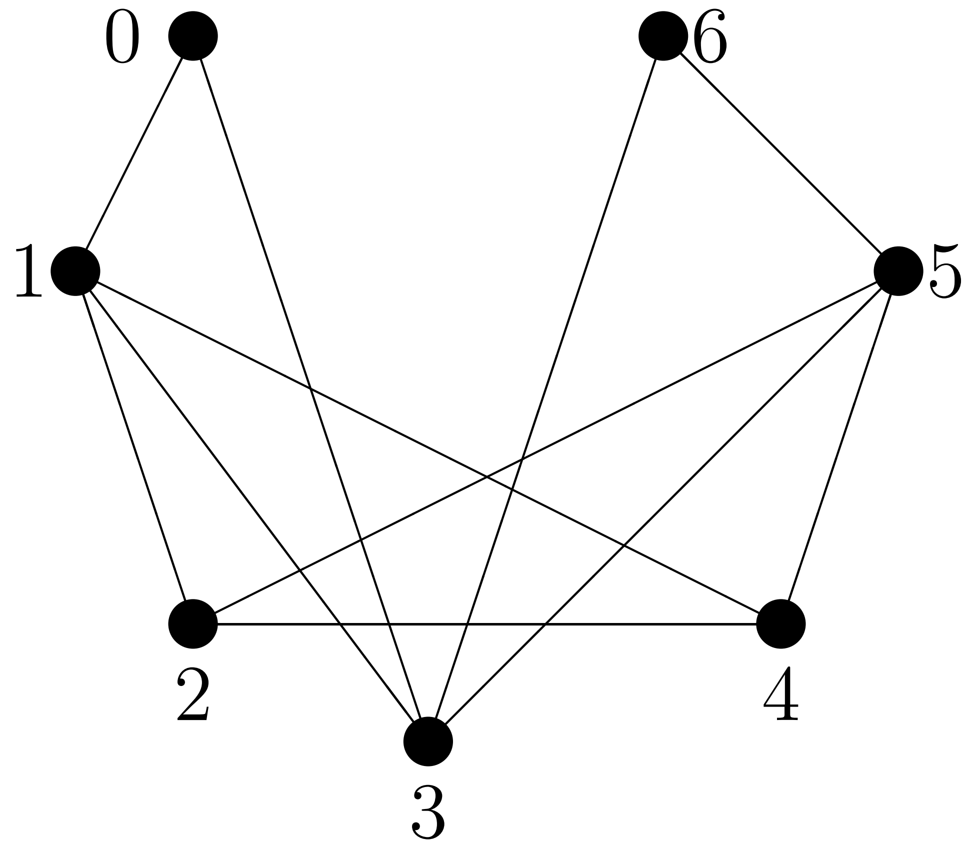
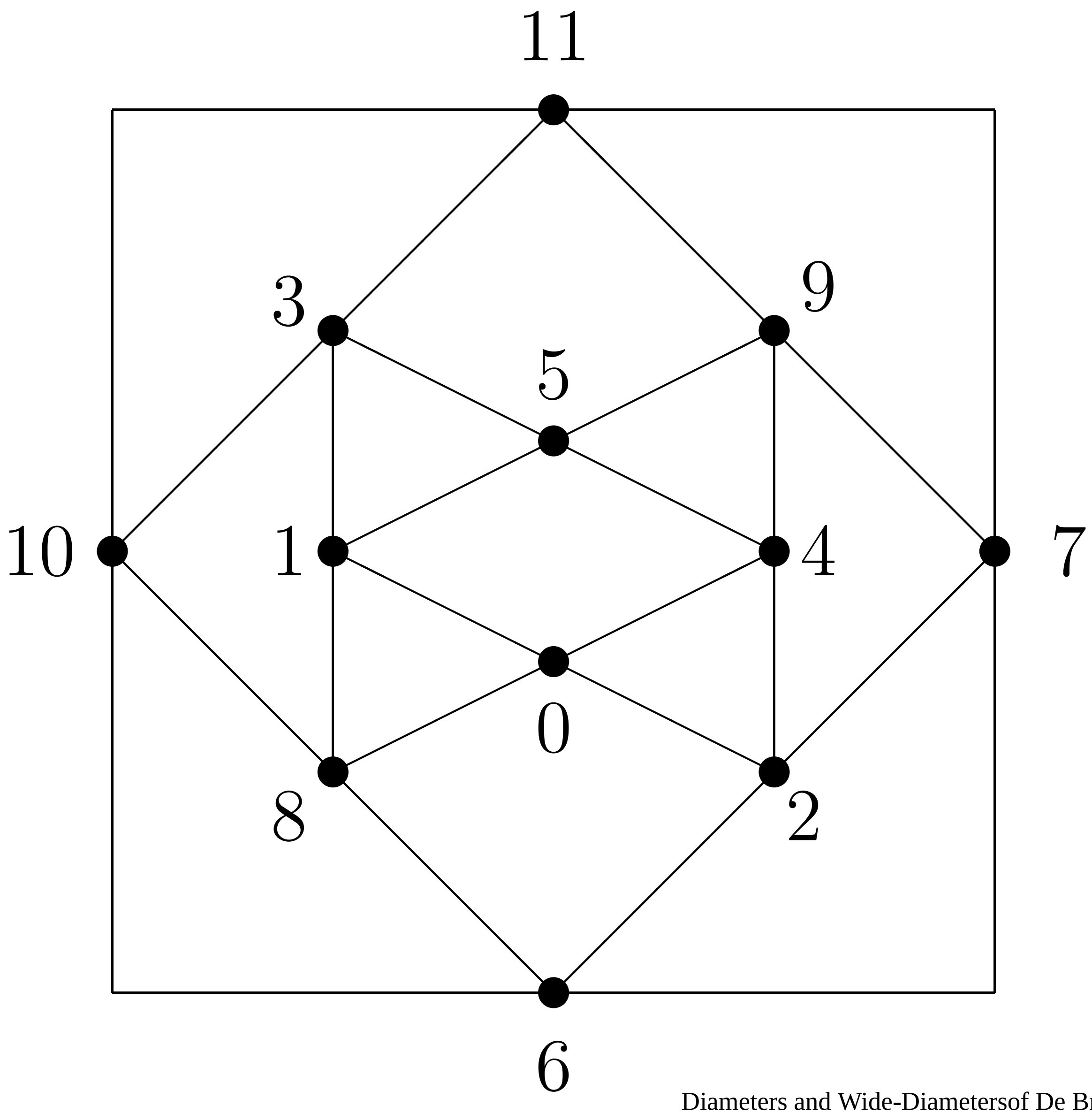


Figure $UG_B(2,7)$



$UG_B(3,12)$ regular Caro '96

$$i \to in + \alpha \pmod{m}, \alpha \not\equiv i \pmod{n}$$



Generalization

Hasunuma and Shibata, 1997

$$B(d,n) \cong G_B(d,d^n)$$

$$UB(d,n) \cong UG_B(d,d^n)$$

The Known Results I

- The connectivity of $B(d, n)$ is $d - 1$.
  Sridhar, 1988

- The connectivity of $UB(d, n)$ is $2d - 2$.
  Esfahnian and Hakimi, 1985

- By Menger's Theorem, there are $2d - 2$ vertices disjoint paths between any two vertices in $UB(d, n)$.

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$(2d-2)$-wide-diameter of $UB(d,n)$?

The Main Theorem I

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$(2d-2)$-wide-diameter of $UB(d,n)$

$$d_w(G) \leq 2n+1$$

The Main Theorem I

$(2d-2)$-wide-diameter of $UB(d,n)$

$$d_w(G) \leq 2n+1$$

For $y, z \in UB(d,n)$, there are $2d-2$ internally vertex-disjoint paths with length at most $2n+1$.

Proof 1/5

$$P[000000ay_1y_2y_3y_4y_5y_6], a \in Z_d^*$$

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$$P[000000ay_1y_2y_3y_4y_5y_6], a \in Z_d^*$$

$$\mathbf{0} \to 00000a \to 0000ay_1 \to 000ay_1y_2 \to 00ay_1y_2y_3$$
$$\to 0ay_1y_2y_3y_4 \to ay_1y_2y_3y_4y_5 \to y_1y_2y_3y_4y_5y_6 = \mathbf{y}$$

Proof 1/5

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Proof 1/5

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$$P[000000az_1z_2z_3z_4z_5z_6], a \in Z_d^*$$

$$\mathbf{0} \rightarrow 00000a \rightarrow 0000az_1 \rightarrow 000az_1z_2 \rightarrow 00az_1z_2z_3$$
$$\rightarrow 0az_1z_2z_3z_4 \rightarrow az_1z_2z_3z_4z_5 \rightarrow z_1z_2z_3z_4z_5z_6 = \mathbf{z}$$

Proof 2/5

Proof 2/5

$$P[y_1 y_2 y_3 y_4 y_5 y_6 b000000], b \in Z_d^*$$

Proof 2/5

$$P[y_1y_2y_3y_4y_5y_6b000000], b \in Z_d^*$$

$$P[z_1z_2z_3z_4z_5z_6b000000], z_i, b \in Z_d^*$$

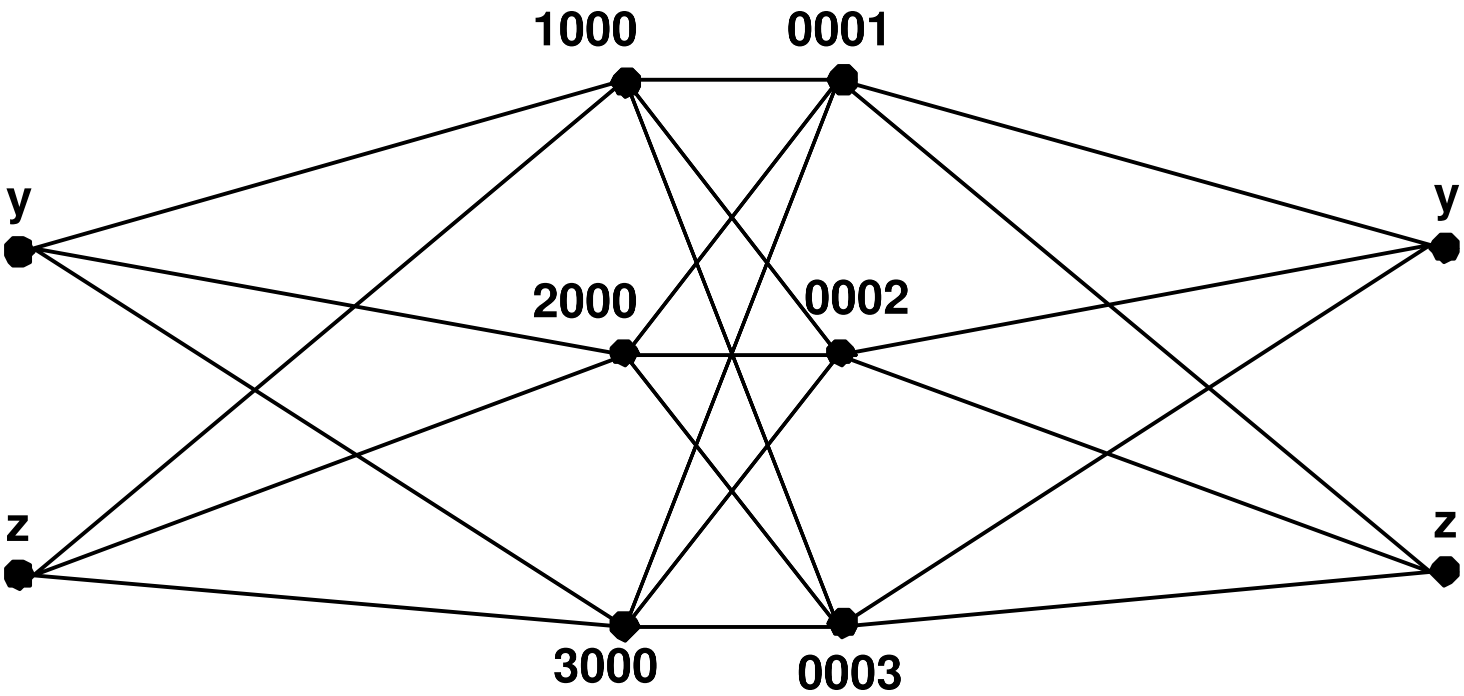
Proof 2/5

$$P[y_1y_2y_3y_4y_5y_6b000000], b \in Z_d^*$$

$$P[z_1z_2z_3z_4z_5z_6b000000], z_i, b \in Z_d^*$$

The proof is the same as $\mathbf{y} \rightarrow \mathbf{0}$

Proof 3/5



Proof 4/5

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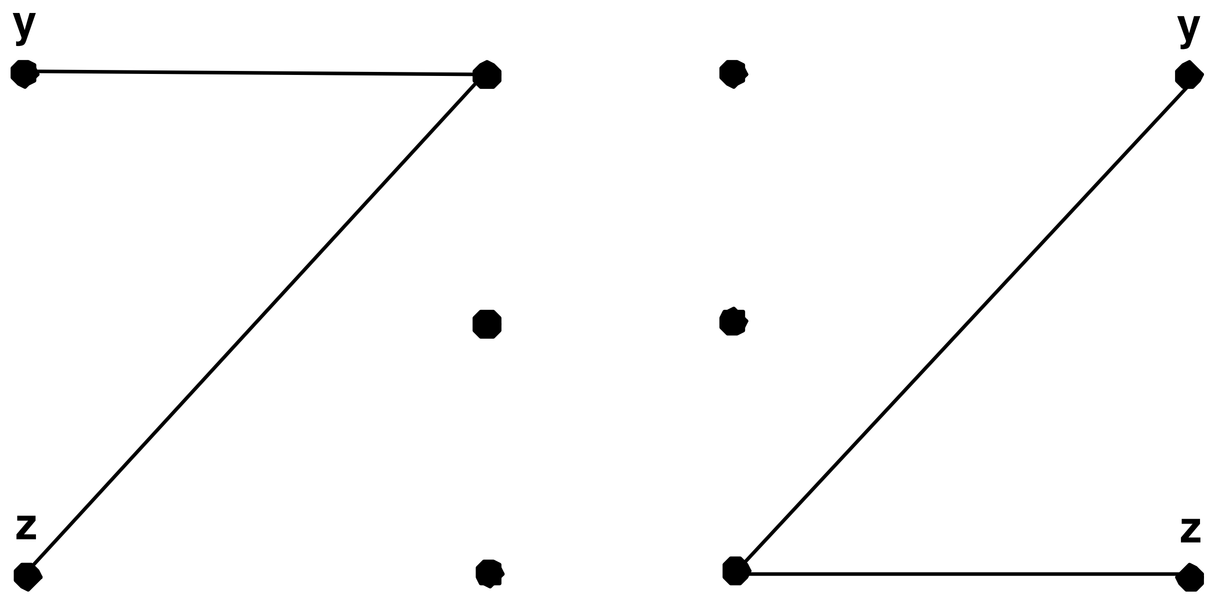
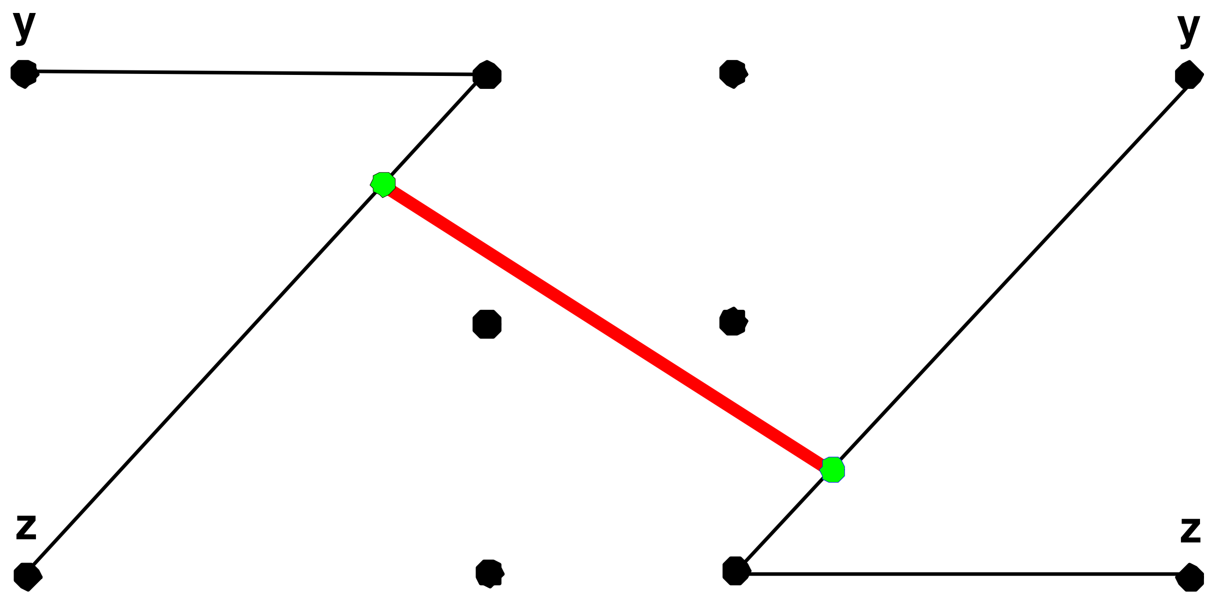
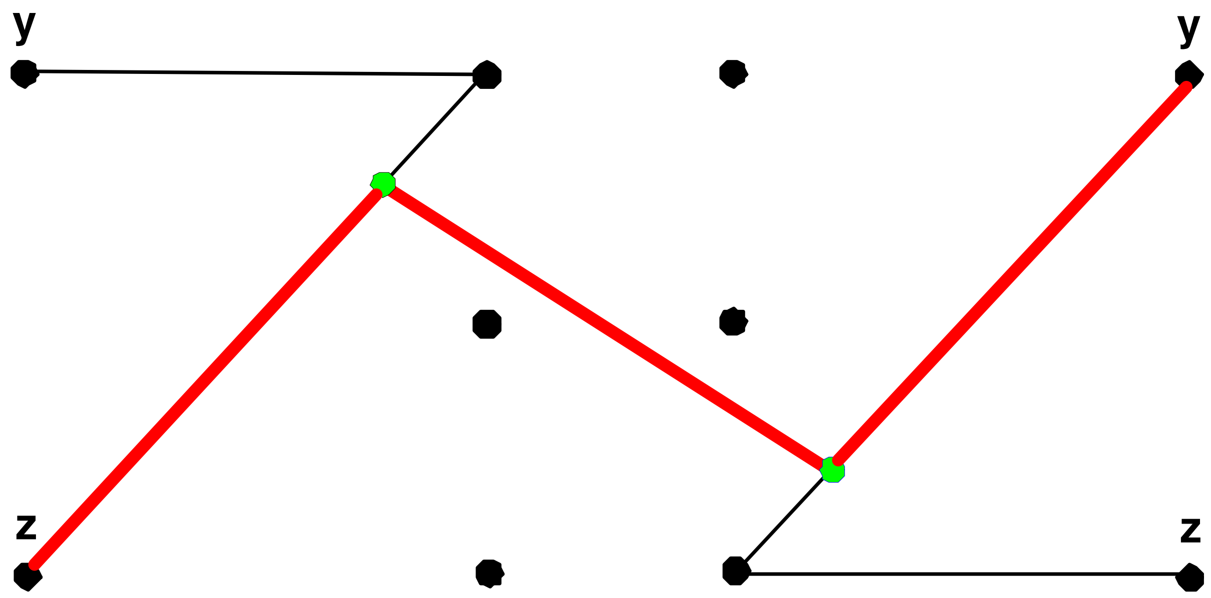


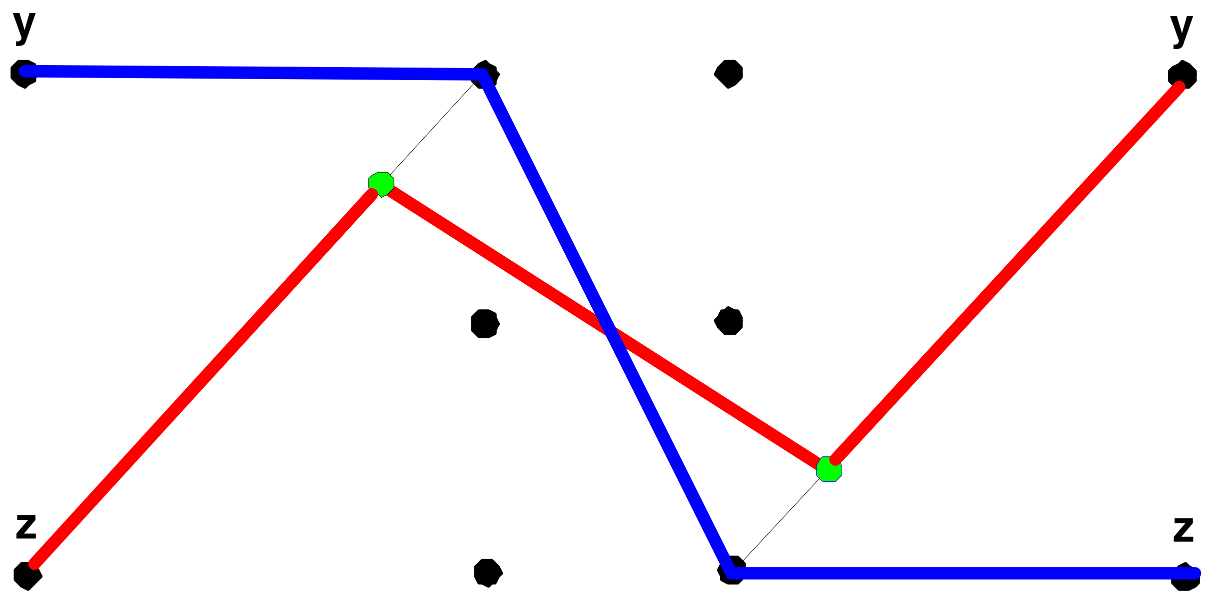
Proof 4/5



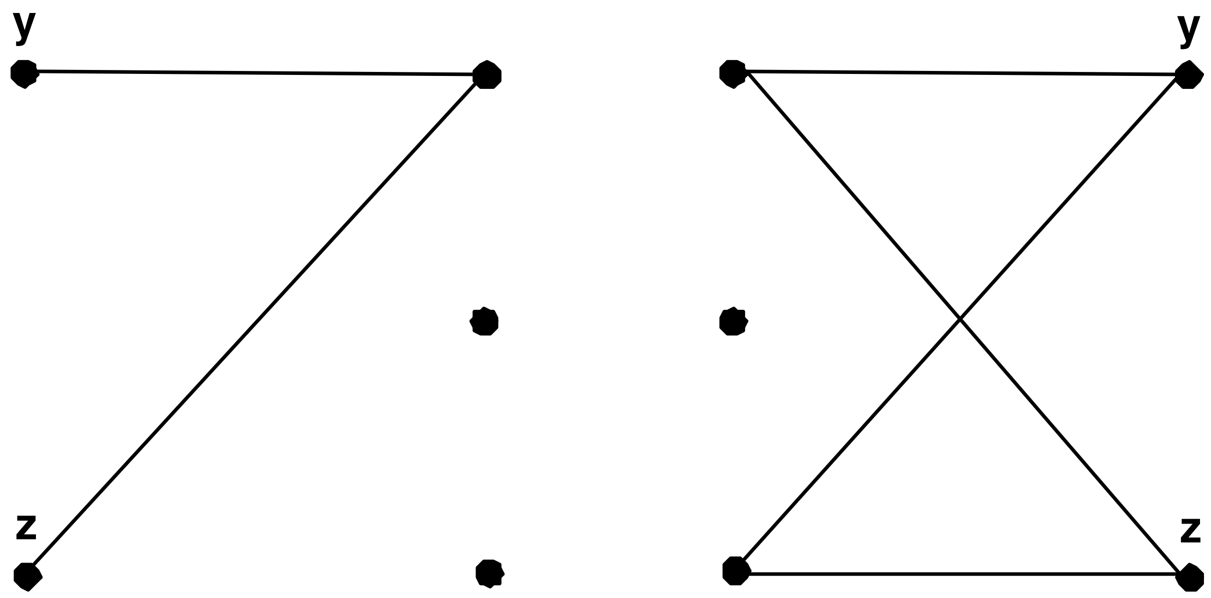
Proof 4/5



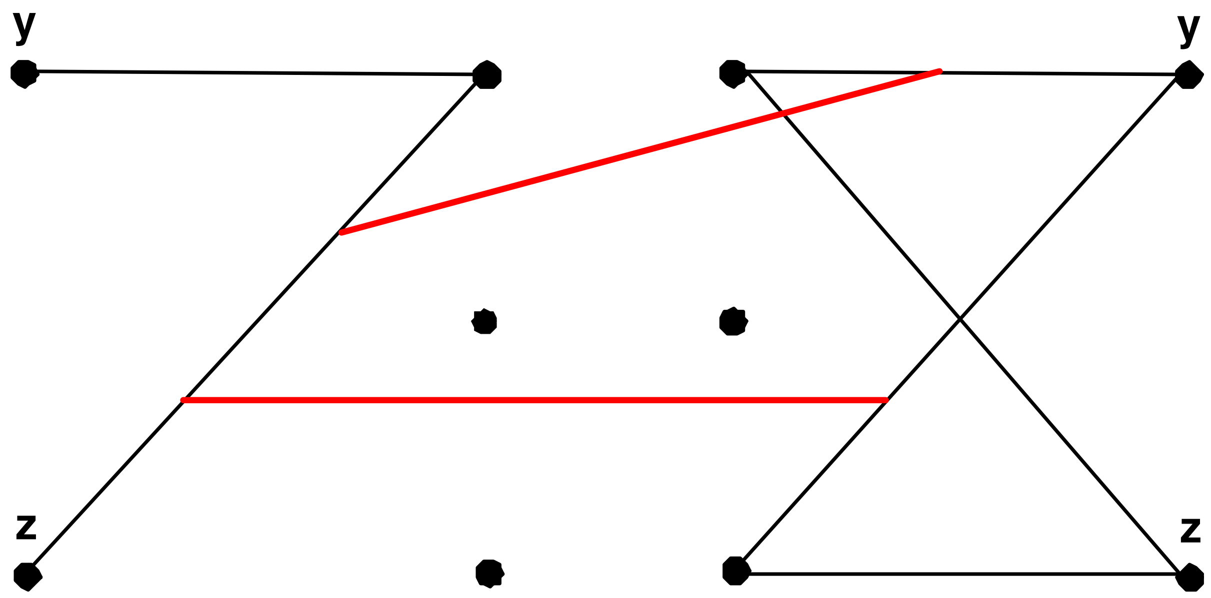
Proof 4/5



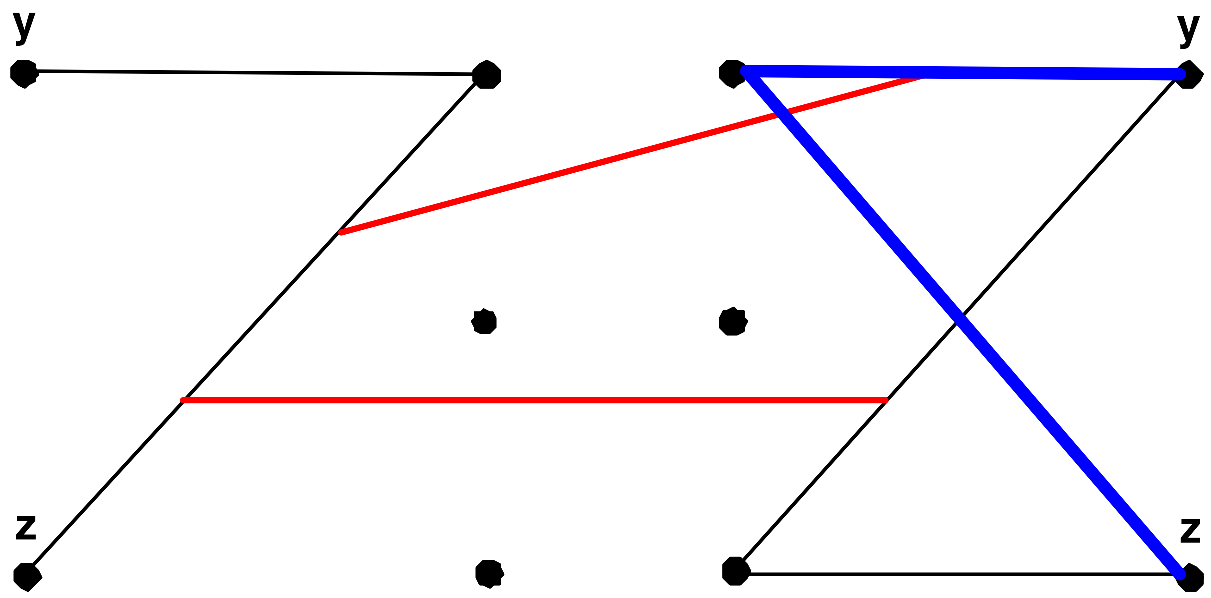
Proof 5/5



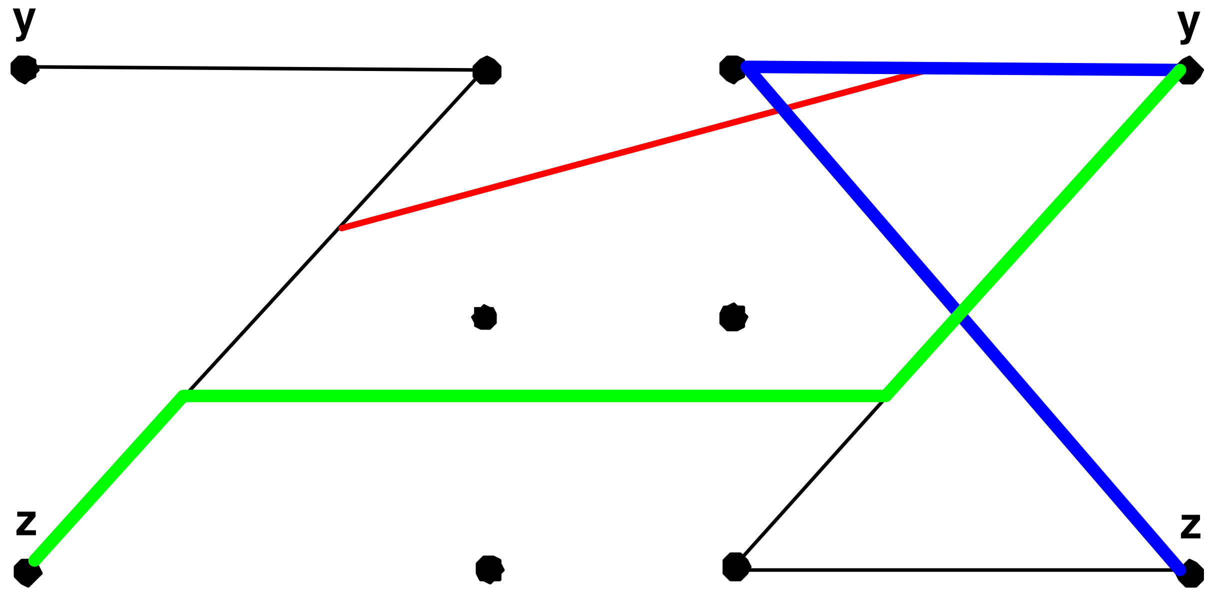
Proof 5/5



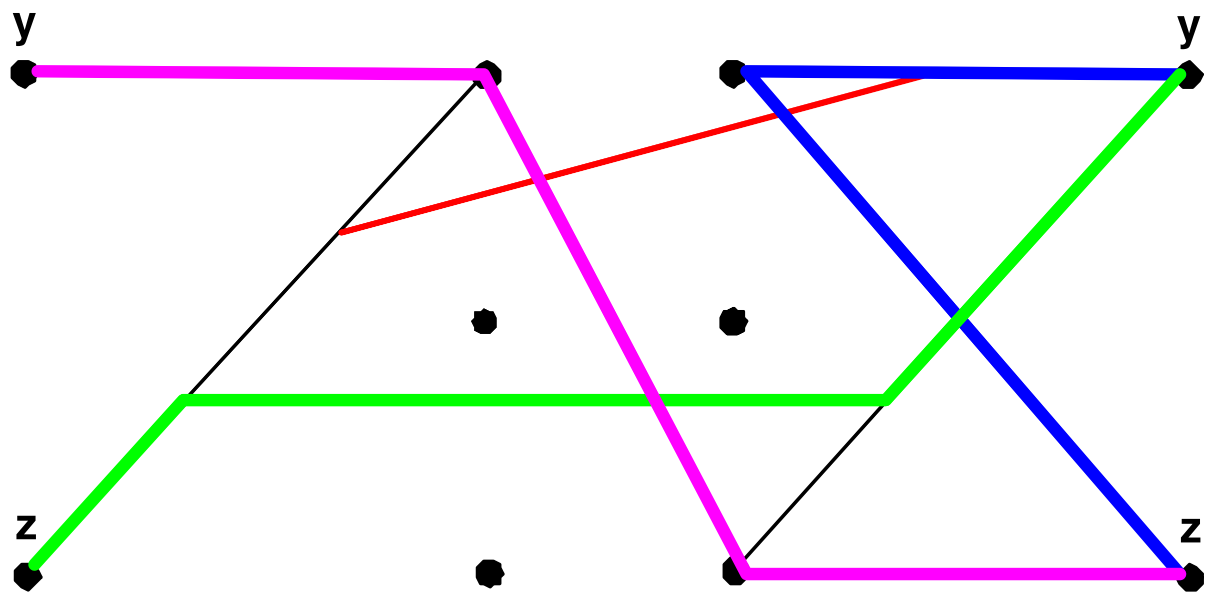
Proof 5/5



Proof 5/5



Proof 5/5



Definition: $UG_B(n, m)$

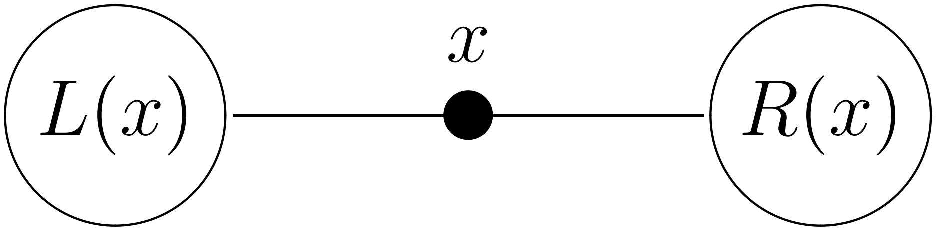
$$i \to in + \alpha \pmod m, \alpha \in [0, n-1]$$

$$N(i) = R(i) \cup L(i)$$

$$R(i) = \{in + \alpha \pmod m | \alpha \in [0, n-1]\}$$

$$L(i) = \{j : jn + \beta \equiv i \pmod m,$$

where $\beta \in [0, n-1], j \in [0, m-1]\}$



The Known Results II, IEEE

- connectivity of $G_B(n,m)$ is $n-1$, Imase, Soneoka, and Okada, 1985
- Imase, Soneoka and Okada, $d(G_B(n,m)) \leq \lceil \log_n m \rceil$, 1981
- Nochefranca and Sy, $d(UG_B(n, n(n+1))) = 3$, 1998
- Escuadro, $d(UG_B(n, n^2)) = 2$, 1997
- Nochefranca, $d(UG_B(n, n(n^2+1))) = 4$ for odd $n \geq 3$, 1997
- Caro, $d(G_B(n,m)) = 3$ for $n^2 \leq m \leq n^3$ and $n|m$, 2002

diameter of $UG_B(n, m)$

$d(G_B(n, m)) \leq \lceil \log_n m \rceil$

Imase et al, 81', $d(G_B(n, m)) \leq \lceil \log_n m \rceil$

$$\lceil \log_n m \rceil \leq \lceil \log_n n^3 \rceil = 3$$

The Main Results II

$d(UG_B(n,m)) = 3$ for

- $2n^2 \le m \le n^3$
- $n^2 + 2n \le m \le 2n^2$
- $n^2 + (\frac{\sqrt{5}+1}{2})n \le m \le n^2 + 2n$
- $m = n^2 + 1$

$d(UG_B(n,m)) = 2$ for $m = n^2 + 2$ or $n^2 + n + 1$

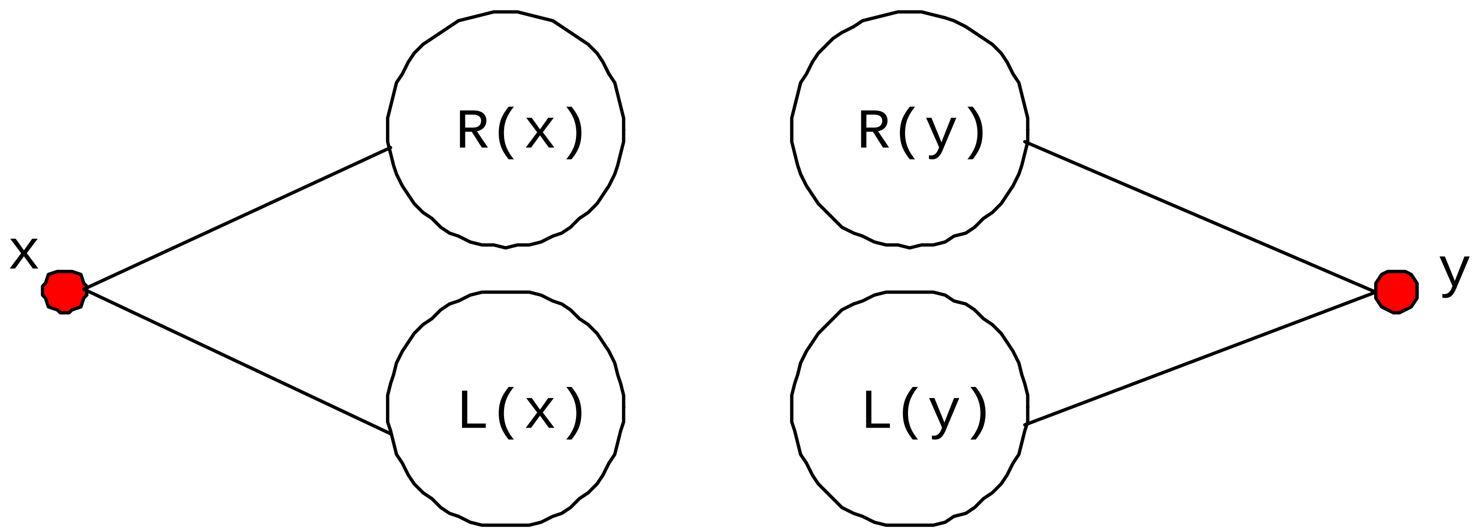
The Main Idea of Proofs

Find $x, y \in [0, m-1]$ such that

$$d_G(x, y) \geq 3$$

$$N(x) \cap N(y) = \emptyset$$

$$(R(x) \cup L(x)) \cap (R(y) \cup L(y)) = \emptyset$$

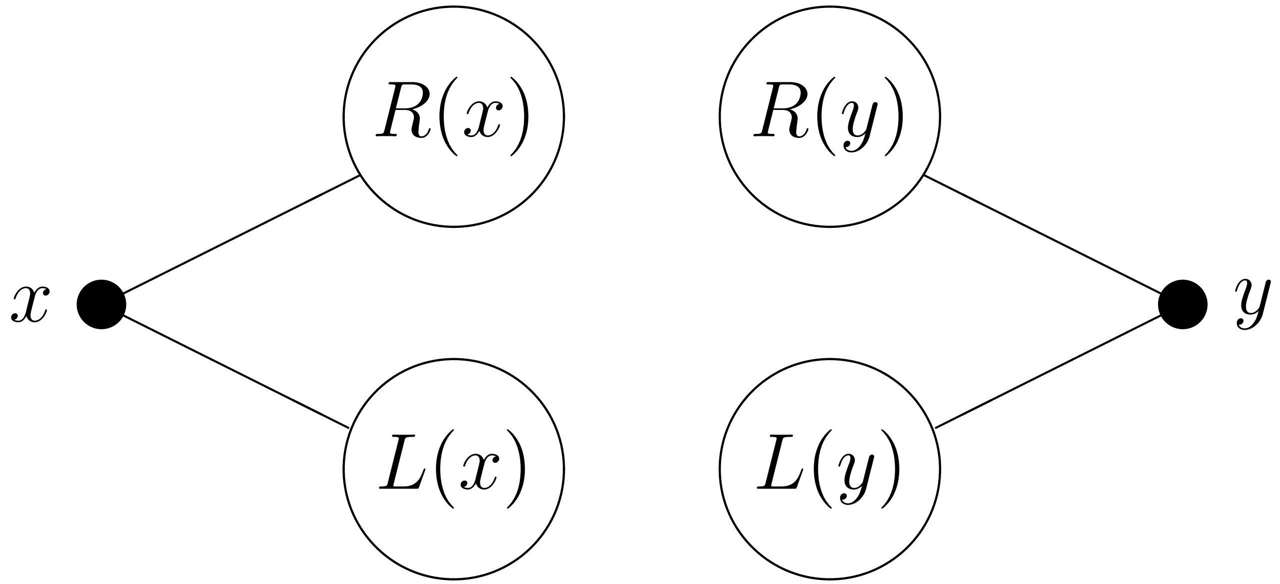


$d_G(x, y) \geq 3$

There exists $x, y \in [0, m-1]$

- $x \notin R(y)$
- $x \notin L(y)$
- $R(x) \cap L(y) = \emptyset$
- $R(x) \cap R(y) = \emptyset$
- $L(x) \cap R(y) = \emptyset$
- $L(x) \cap L(y) = \emptyset$

Figure of Main Idea



Key Points of Proofs

$2n^2 \le m \le n^3,$

$$x = 0, y \in \{m - n, m - n - 1\}$$

$n^2 + 2n \le m \le 2n^2,$

$$x = 1, y \in [n^2 - 2n, n^2 - 1]$$

$n^2 + (\frac{\sqrt{5}+1}{2})n \le m \le n^2 + 2n,$

$$x = 0, y \in [n^2, m - n]$$

$m = n^2 + 1$

$$x = n - 1, y = n^2 - n + 2$$

Theorem $d_G(x, y) = 3$

For $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq n^2 + 2n, n \geq 3$

$$d_G(x, y) \leq \lceil \log_n n^3 \rceil \leq 3$$

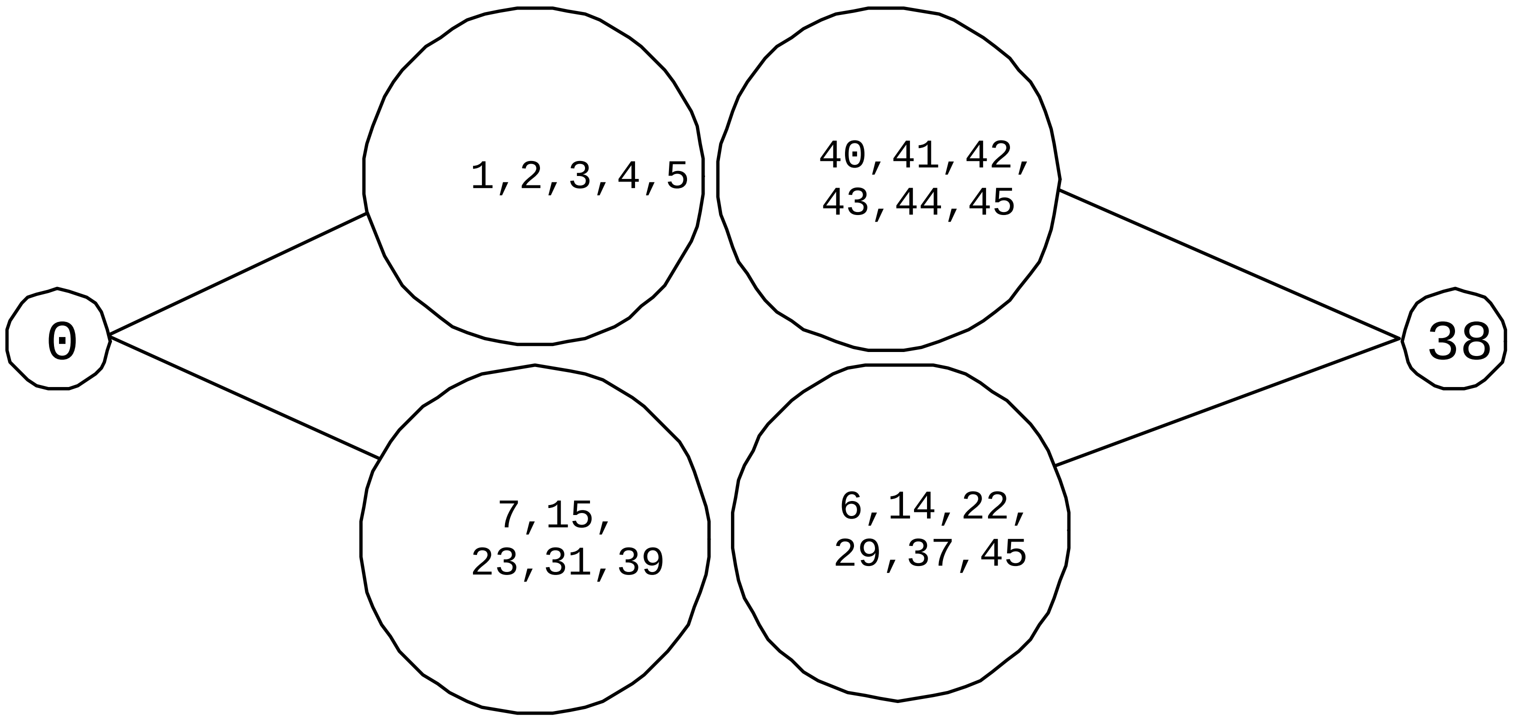
$$3 \leq d_G(x, y)$$

when $x = 0, y \in Y = [n^2, m - n]$

Then

$$d_G(x, y) = 3$$

Example: n=6, m=47



Proof 1/6

(1)$\forall y \in Y, 0 \notin L(y)$

$$R(0) = [1, n-1]$$

$$(Y = [n^2, m-n]) \cap [1, n-1] = \emptyset$$

Proof 2/6

$(2) \forall y \in Y, R(0) \cap L(y) = \emptyset$

$$\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1, n-1]} R(i) = [n, n^2 - 1].$$

$$[n, n^2 - 1] \cap ([n^2, m - n] = Y) = \emptyset$$

Proof 3/6

(3)$\forall y \in Y, L(0) \cap L(y) = \emptyset$

Suppose not. There exists $k \in [0, m-1]$

$$\begin{cases} kn + \alpha \equiv 0 \pmod{m}, \\ kn + \beta \equiv y \pmod{m}. \end{cases}$$

$$\beta - \alpha \equiv y \pmod{m}$$

$$\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$$

$$Y = [n^2, m-n]$$

A contradiction.

Proof 4.1/6

$\exists y_0 \in Y, L(0) \cap R(y_0) = \emptyset$

$$0 \notin \bigcup_{i \in R(y_0)} R(i) \Leftrightarrow y_0 n^2 \in [1, m - n^2]$$

Let $A_i = [(i-1)(m-n^2)+1, i(m-n^2)]$,
$i = 1, 2, \ldots, t+1, t = m - n^2 - n$

$$|\bigcup_{i=1}^{t+1} A_i| = (t+1)(m-n^2) = (t+1)(n+t) \geq m$$

$\exists i_0$, such that $n^4 \in A_{i_0}$.

Proof 4.2/6

Let $y_0 = n^2 + i_0 - 1$. Then

$$y_0 n^2 \equiv (n^2 + i_0 - 1)n^2 \equiv n^4 + (i_0 - 1)n^2$$

$$y_0 n^2 \in [1, m - n^2]$$

$$\bigcup_{i \in R(y_0)} R(i) = [1, m - n^2 + n^2 - 1] = [1, m - 1]$$

$$0 \notin \bigcup_{i \in R(y_0)} R(i)$$

$$L(0) \cap R(y_0) = \emptyset$$

Proof 5/6

For y in (4), $R(0) \cap R(y) = \emptyset$

Suppose not. $yn + \alpha \equiv 0n + \beta$

$$yn \equiv \beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$$

$$yn^2 \in \{0, n, \ldots, (n-1)n\} \cup \{2n+t, \ldots, n^2+t\}.$$

$$yn^2 \in [1, m-n^2] = [1, n+t], \text{ since } (4)$$

$$yn^2 \equiv n$$

$$yn \equiv 1$$

There are no solutions. A contradiction.

Proof 6/6

For y in (4), $0 \notin R(y)$
Suppose not. $yn + \alpha \equiv 0$
Case $\alpha = 0$

$$yn^2 \equiv (yn)n \equiv 0$$

$$yn \in L(0) \cap R(y)$$

A contradiction to (4).
Case $\alpha \neq 0$

$$yn^2 \equiv -\alpha n \notin [1, m - n^2]$$

A contradiction to (4).

Lemma $d(UG_B(n, n^2 + 2)) = 2$

$$\forall x \neq y, d_G(x, y) \leq 2$$

$$N(x) \cap N(y) \neq \emptyset$$

$$(1) R(x) \cap L(y) \neq \emptyset$$

$$(2) R(y) \cap L(x) \neq \emptyset$$

$$(3) R(x) \cap R(y) \neq \emptyset$$

$$(4) L(x) \cap L(y) \neq \emptyset$$

one of the 4 conditions holds.

Proof 1/3 $m = n^2 + 2$

$$R(x) \cap L(y) \neq \emptyset$$

$$(nx + \alpha)n + \beta \equiv y$$

$$y + 2x \equiv \alpha n + \beta \in [0, n^2 - 1]$$

$$\Rightarrow y + 2x \equiv \{n^2, n^2 + 1\}$$

$$y+2x \in \{n^2, n^2+1, 2n^2+2, 2n^2+3, 3n^2+4, 3n^2+5\}$$

Proof 2/3 $m = n^2 + 2$

$$R(y) \cap L(x) \neq \emptyset$$

$$(ny + \alpha)n + \beta \equiv x$$

$$x + 2y \equiv \alpha n + \beta \in [0, n^2 - 1]$$

$$\Rightarrow x + 2y \equiv \{n^2, n^2 + 1\}$$

$$x + 2y \in \{n^2, n^2 + 1, 2n^2 + 2, 2n^2 + 3, 3n^2 + 4, 3n^2 + 5\}$$

Proof 3/3 $m = n^2 + 2$

$$y + 2x \notin [0, n^2 - 1]$$

$$\Rightarrow y + 2x \in \{n^2, n^2 + 1\}$$

$$x + 2y \notin [0, n^2 - 1]$$

$$\Rightarrow x + 2y \in \{n^2, n^2 + 1\}$$

Then

$$x + 2y \in \{n^2, n^2 + 1, 2n^2 + 2, 2n^2 + 3\}$$

$$y + 2x \in \{n^2, n^2 + 1, 2n^2 + 2, 2n^2 + 3\}$$

Conclusions

$(2d-2)$-wide-diameter of $UB(d,n) \leq 2n+1$
J. Combin. Optimation, 14 (2007), 143-152

diameter of $UG_B(n,m)$ for $2n^2 \leq m \leq n^3$
Networks, to appear

diameter of $UG_B(n,m)$ for $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq 2n^2$
Ars Combinatoria, to appear

Future Work

Let $G = UG_B(n,m)$

- When $n^2 < m < n^2 + (\frac{\sqrt{5}+1}{2})n$, diameter of G?
- When $m > n^3$, diameter of G?
- connectivity of G?
- wide diameters of G?

Thanks

Theorem $d_G(x, y) = 3$

For $2n^2 \leq m \leq n^3$

$$d_G(x, y) \leq \lceil \log_n n^3 \rceil \leq 3$$

$$3 \leq d_G(x, y)$$

when $x = 0, y = m - n$ or $m - n - 1$

Then

$$d_G(x, y) = 3$$

Proof $2n^2 \le m \le n^3$ 1/5

We claim

$$d_G(0, m-n) \ge 3 \text{ or}$$

$$d_G(0, m-n-1) \ge 3.$$

Let

$$j_1 = m - n,$$

$$j_2 = m - n - 1, \text{ and}$$

$$j = j_1 \text{ or } j_2.$$

Proof $2n^2 \leq m \leq n^3$ 2/5

(1) $R(0) \cap L(j) = \emptyset$

$$\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1, n-1]} R(i) = [n, n^2 - 1]$$

$$n^2 - 1 < 2n^2 - n \leq m - j$$

Proof $2n^2 \leq m \leq n^3$ 3/5

(2) $R(0) \cap R(j) = \emptyset$

$$R(j) = \{jn + \alpha : \alpha \in [0, n-1]\}$$

$$R(j) = \{m - n^2 + \alpha \text{ or } m - n^2 - n + \alpha\}$$

$$= [n^2 - n, m - n^2 - 1]$$

$$R(0) = [1, n-1]$$

Proof $2n^2 \le m \le n^3$ **4/5**

(3) $L(0) \cap L(j) = \emptyset$

$$\begin{cases} kn + \alpha \equiv 0 \pmod{m}, \\ kn + \beta \equiv j \pmod{m}. \end{cases}$$

$$\beta - \alpha \equiv j \pmod{m}$$

$$\beta - \alpha = [0, n-1] \cup [m-n+1, m-1]$$

Proof $2n^2 \leq m \leq n^3$ 5/5

(4) $L(0) \cap R(j) = \emptyset$

$$j = j_1 \text{ or } j_2$$

Long time to proof.

Theorem $d_G(x, y) = 3$

For $n^2 + 2n \leq m \leq 2n^2$

$$d_G(x, y) \leq \lceil \log_n n^3 \rceil \leq 3$$

$$3 \leq d_G(x, y)$$

$$\text{when } x = 1, y \in Y = [n^2 - 2n, n^2 - 1]$$

Then

$$d_G(x, y) = 3$$

Proof $n^2 + 2n \leq m \leq 2n^2$ **1/7**

$$x = 1, y \in Y = [n^2 - 2n, n^2 - 1]$$

Proof $n^2 + 2n \leq m \leq 2n^2$ **2/7**

(1) For each $y \in Y$, $1 \notin L(y)$

$$R(1) = [n, 2n - 1]$$

Proof $n^2 + 2n \leq m \leq 2n^2$ 3/7

(2) $R(1) \cap L(y) = \emptyset$

$$\bigcup_{i \in R(1)} R(i) = [n^2, 2n^2 - 1]$$

$$= [n^2, m-1] \ \cup \ [0, 2n^2 - 1 - m].$$

Proof $n^2 + 2n \leq m \leq 2n^2$ 4/7

(3) $L(1) \cap L(y) = \emptyset$

$$\begin{cases} kn + \alpha \equiv 1 \pmod{m}, \\ kn + \beta \equiv j \pmod{m}. \end{cases}$$

$$\beta - \alpha \equiv j - 1 \pmod{m}$$

$$\beta - \alpha = [0, n-1] \cup [m - n + 1, m - 1]$$

Proof $n^2 + 2n \leq m \leq 2n^2$ **5/7**

(4) $R(1) \cap R(y) = \emptyset$ There exists a set of at most four elements, Y', such that for each $y \in Y \setminus Y'$, $R(1) \cap R(y) = \emptyset$

Proof $n^2 + 2n \leq m \leq 2n^2$ 6/7

(5) $L(1) \cap R(y) = \emptyset$ There exist at least three elements y in Y such that $L(1) \cap R(y) = \emptyset$.

Proof $n^2 + 2n \leq m \leq 2n^2$ **7/7**

(6) $1 \notin R(y)$ For each $y \in Y$ satisfying (4) and (5), $1 \notin R(y)$.

Lemma $m = n^2 + 3$

If n is an even positive integer or $n = 8p + 5$, where $p \geq 0$ is an integer, then $d(UG_B(n, n^2 + 3)) = 2$.

Lemma $m = n^2 + t, t \geq 4$

If $t, p \in N$, $t \geq 4$, $p \geq 2$, $n = p(t-1)$, and $m = n^2 + t$, then $d(UG_B(n, m)) = 2$.

Lemma $n^2+n+3 \leq m \leq n^2+\frac{3}{2}n$

$diam(UG_B(n,m)) = 3$ for $n^2+n+3 \leq m \leq n^2+\frac{3}{2}n$ and $n \geq 3$.

Lemma

$diam(UG_B(n,m)) = 3$ for $n^2 + \frac{3}{2}n < m \leq n^2 + (\frac{\sqrt{5}+1}{2})n$ and $n \geq 3$.

conclusions

solved:

$$n^2 \le m \le n^2 + 3,$$

$$n^2 + n \le m \le n^3$$

unsolved:

$$n^2 + 4 \le m < n^2 + n$$