陳巧玲

2. Let A be an infinite set. Prove that for each $k(\geq 1)$ coloring of the elements in $\binom{A}{3}$, there exists a monochromatic $\binom{T}{3}$ where T is also an infinite set.

By induction on r

r=1 : clearly true.

Suppose r<k the assertion is true (r ≥ 2)

Now, consider r=k:

Let φ be a k-coloring of $\binom{A}{r}$ and $A_1 = A_0$.

Let $x_1 \in A_0$ and $B_1 = A_0 \setminus \{x_1\}$

$\forall S_1 \in \binom{B_1}{r-1}$

Define $\varphi^{(1)}(S_1) = \varphi(S_1 \cup \{x_1\})$

Then, since $S_1 \cup \{x_1\}$ is r-set in A and $\varphi^{(1)}(S_1) = \varphi(S_1 \cup \{x_1\})$

$\varphi^{(1)}$ is k-coloring of $\binom{B_1}{r-1}$.

So, by induction , there exists A_1 in B_1, s.t $\binom{A_1}{r-1}$ is monotonic.

Similaly, let $x_2 \in A_1$ and $B_2 = A_1 \setminus \{x_2\}$

$\forall S_2 \in \binom{B_2}{r-1}$

Define $\varphi^{(2)}(S_2) = \varphi(S_2 \cup \{x_2\})$

Then, since $S_2 \cup \{x_2\}$ is r-set in A, and $\varphi^{(2)}(S_2) = \varphi(S_2 \cup \{x_2\}), \varphi^{(2)}$ is k-coloring of $\binom{B_2}{r-1}$.

So, by induction , there exists A_2 in B_2, s.t $\binom{A_2}{r-1}$ is monotonic.

Using the same process , we have that $\binom{A_i}{r-1}$ is monotonic $i \in N$.

Furthermore, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

Since we only use k colors , by pigeonhole principle, there exists one colors which occurs in $\binom{A_{i_1}}{r-1}, \binom{A_{i_2}}{r-1}, \cdots$, (inf inite many),where $A_{i_1} \supseteq A_{i_2} \supseteq A_{i_3} \supseteq \cdots$

Let T= $\left\{ x_{i_1} \quad x_{i_2} \quad x_{i_3} \quad \cdots \right\}$ T is infinite.

Then , we have the following results.

(1)

Since we use the same color in $\binom{A_{i_1}}{r-1}, \binom{A_{i_2}}{r-1}, \cdots$, and $A_{i_1} \supseteq A_{i_2} \supseteq A_{i_3} \supseteq \cdots$,

and r-1 set of T are the same color.

(2)

For any r-set $\left\{ x_{i_{c_1}}, \quad x_{i_{c_2}}, \quad x_{i_{c_3}}, \quad \cdots x_{i_{c_{r-1}}}, \quad x_{i_{c_r}} \right\}$ of T, $c_1 < c_2 < \cdots < c_r$,

$\varphi(\left\{ x_{i_{c_1}} \right\} \cup \left\{ x_{i_{c_2}}, \quad x_{i_{c_3}}, \quad \cdots \quad x_{i_{c_r}} \right\}) = \varphi^{(i_{c_1})}(\left\{ x_{i_{c_2}}, \quad x_{i_{c_3}}, \quad \cdots \quad x_{i_{c_r}} \right\})$, where $\left\{ x_{i_{c_2}}, \quad x_{i_{c_3}}, \quad \cdots \quad x_{i_{c_r}} \right\} \in \binom{A_{i_{c_2}}}{r-1}$.

Thus, by (1), (2) ,any r-set of are the same color

$\rightarrow \binom{T}{r}$ is monotonic, where T is infinite.

劉士慶

3. Find a deterministic algorithm to prove that the domination number of a graph G $D(G) \le \frac{n(1+\ln\delta(G)+1)}{\delta(G)++1}$.

Let G be a graph with $\delta(G)$=k. Given S$\subseteq$V(G).

Let U be the set of vertices not dominated by S.

Claim: There exists some vertex y$\in$V (G) and y$\notin$S such that y dominates at least $\frac{|U|(k+1)}{n}$ vertices of U.

Proof of claim: $\because \delta(G)=k$ and $U \subseteq V(G)$ $\therefore \sum_{v\in U}|N[v]| \ \ge \ |U|(k+1)$.

$\because \Delta(G)=n$. $\therefore \frac{\sum_{v\in U}|N[v]|}{|G \setminus S|} \ge \frac{|U|(k+1)}{n}$.

$\frac{\sum_{v\in U}|N[v]|}{|G \setminus S|}$:平均 each 在 G\S 的 vertex 會落在幾個 N[v]裡($v \in U$)

Hence there exists some vertex $y \in V(G)(y \notin S)$ appears at least $\frac{|U|(k+1)}{n}$ times.

Method: We select a vertex repeatly that dominates the most of the remaining undominated vertices.

Note: When r undominated vertices remain, after one step of selection, there remain at most $r(1-\frac{k+1}{n})$ undominated vertices.

After $\frac{n*\ln(k+1)}{k+1}$ steps (from S =$\varnothing$), the number of undominated is

$$n(1-\frac{k+1}{n})^{\frac{n*\ln(k+1)}{k+1}} < n\,[e^{(\frac{-(k+1)}{n})}]^{\frac{n*\ln(k+1)}{k+1}} \; (\because 1-p<e^{-p})$$

$$=n*e^{-\ln(k+1)}=\frac{n}{k+1}$$

The number of vertices of a dominating set=(number of vertices we select)+(number of undominated vertices).

Hence after $\frac{n * \ln(k+1)}{k+1}$ steps, we form a dominating set of size at most

$$\frac{n * \ln(k+1)}{k+1} + \frac{n}{k+1} = \frac{n(1+\ln(k+1))}{k+1} = \frac{n(1+\ln \delta(G)+1)}{\delta(G)++1}.$$

4. Prove that almost all graphs are 100-connected.

Proof. .

Claim 1 : For every constant $p \in (0,1)$ and $i, j \in \mathbb{N}$, almost all graphs G^p has the property $P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U| = i$ and $|V| = j$, a vertex $v \notin U \cup W$ that is adjacent to all the vertices of U but to none of vertices in W

Proof of claim 1

The probability that $v \in V(G) \setminus (U \cup W)$ is adjacent to U but not to W is

$$p^i q^j \quad \text{where } q = (1-p)$$

Hence, the probability that no suitable v exists for these U and W is

$$(1 - p^i q^j)^{n-i-j}$$

Since the number of $< U, W >$ pairs is

$$\begin{aligned}
\binom{n}{i}\binom{n-i}{j} &= \frac{n!}{(n-i)!i!}\frac{(n-i)!}{(n-i-j)!j!} \\
&= \frac{n!}{i!(n-i-j)!j!} \\
&\leq \frac{n!}{(n-i-j)!} \\
&\leq n^{i+j}
\end{aligned}$$

So the total probability is less than

$$\begin{aligned}
n^{i+j}(1 - p^i q^j)^{n-i-j} &= n^{i+j}(\frac{1}{k})^{n-i-j} \quad \text{for some } k > 1 \\
&= \frac{n^{i+j}}{k^{n-i-j}} \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

Claim 2 : For every constant $p \in (0, 1)$ and $k \in \mathbb{N}$, almost all graphs are k-connected.

Proof of claim 2

Let $i = 2$ and $j = k - 1$. Let G be a graph with property $P_{i,j}$ and give W be an arbitrary set of V(G) and $|W| = k-1$. Then $\forall x, y \in V(G) \setminus W$, then either x is adjacent to y or x and y have a common neighbor. Therefore, W is not a vertex cut, then G is k-connected. By claim 1,

$$\text{almost graphs } G \text{ has property } P_{i,j} \text{ and } |G| \geq k + 2$$

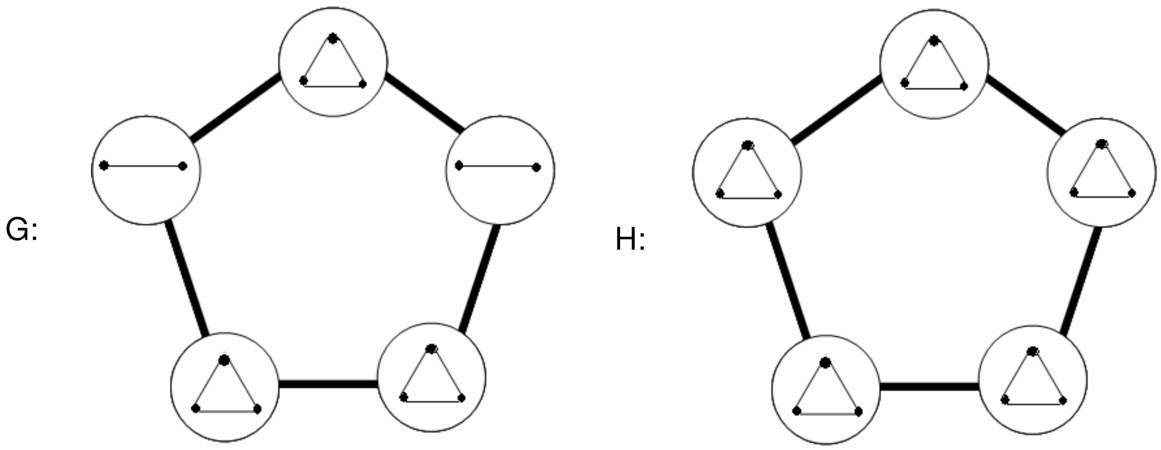
Therefore, almost all graphs are k-connected.

By claim 2, we know almost all graphs are 100-connected.

6. Disprove Hajos conjecture (in vertex coloring). 游舜婷

Hajos Conjecture Every k-chromatic graph contains a subdivision of K_k.

We want to give counterexamples to disprove it. Thick edges below indicate that every vertex in one circle is adjacent to every vertex in the other.

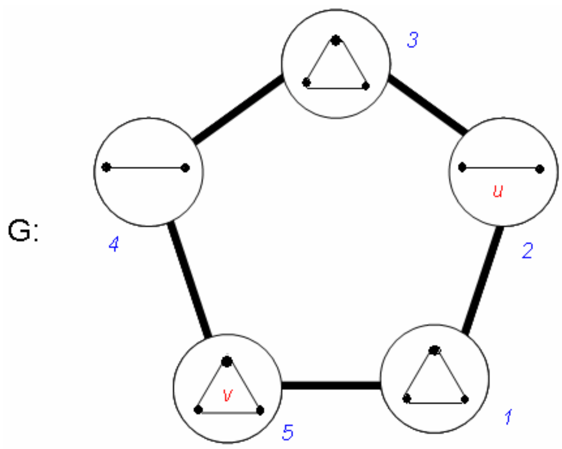


Claim: (1). $\chi(G) = 7$ but G has no K_7-subdivision.
(2). $\chi(H) = 8$ but H has no K_8-subdivision.

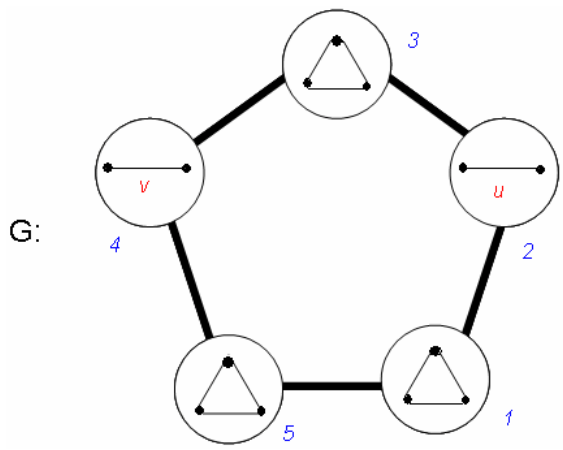
(1). Let $G = G_7 = C_5[K_3, K_2, K_3, K_2, K_3]$.
Because of the construction, we can't take two vertices from the same circle or adjacent circles in an independent set. Hence the independence number $\alpha(G) = 2$. $\Rightarrow \chi(G) \geq \frac{|G|}{\alpha(G)} = \frac{13}{2} = 6.5$. $\Rightarrow \chi(G) \geq 7$.
Since we can use colors $123, 45, 267, 14, 567$ in the successive cliques of G, $\chi(G) \leq 7$. Thus, $\chi(G) = 7$.

Claim: G has no K_7-subdivision.
Suppose G has a K_7-subdivision. Since there are at most six vertices in any two adjacent circles, there exists two vertices u, v of degree $7-1 = 6$ in nonadjacent circles.

(Case1) One of u, v is chosen from a circle of size 3. Since u, v are vertices of degree six in the K_7-subdivision, there must be six pairwise internally disjoint $u - v$ paths in G. This is impossible, since u, v has a separating set of size 5. It contradict to the existence of K_7-subdivision.
Ex:



If u, v are in circles 2 and 5, then the union of circle 1 and 4 is a separating set of u, v of size 5.
(Case2) u, v are chosen from the circles of size 2.



W.L.O.G. assume u is in the circle 2 and v is in the circle 4.
If one of the other five vertices of degree 6 in the K_7-subdivision is in the circle 1 or in the circle 5, then we go back to case1($\rightarrow\leftarrow$).
$\Rightarrow$ The vertices of degree 6 in the K_7-subdivision are the vertices in circle 2,3,4.

$\Rightarrow$ The four internally disjoint paths cinnected the circle 2 and circle 4 must use the circle 1 and 5. (這四條 path 是在K_7上的邊加上 degree 2 的點, 所以一定只能經過 circle 1 and circle 5).
This is impossible, since there are only three internally disjoint paths. Hence G has no K_7-subdivision.
$\Rightarrow$ G is a counterexample of Hajos conjecture.
Use the same argument we have H is also a counterexample of Hajos conjecture.

林逸軒

7. For every positive integer $\mathbf{k \geq 3}$, prove that there exists a graph G such that $\mathbf{g(G)\chi(G) > k^2}$.

Proof: Fix $\varepsilon. 0 < \varepsilon < \frac{1}{k}$.

Let $p = n^{\varepsilon - 1}$.

Let $X(H) \overset{\text{def}}{=}$ the number of cycles of length at most k.

Since the number of cycles of length k' is equal to $\frac{n^{r'}}{2k'}$ where r'=n(n-1)(n-2)…(n-k'+1)..

(有 n(n-1)…(n-k'+1)種選擇 k'-cycle, 但每一個 k'-cycle 有 2k'個一樣的 cycle).

$E(X) = \sum_{k'=3}^{k} \frac{n^{k'}}{2k'} p^{k'} \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i \quad (np = nn^{\varepsilon-1} = n^{\varepsilon} \geq 1)$

$\leq \frac{1}{2} \sum_{i=3}^{k} (np)^k = \frac{1}{2}(k-2)(np)^k$.

By Markov's Inequality

$p\left(X \geq \frac{n}{2}\right) \leq \frac{E(X)}{n/2} \leq (k-2)n^{k-1}p^k = (k-2)n^{k-1}n^{(\varepsilon-1)k} = (k-2)n^{k\varepsilon-1}$.

Since $\varepsilon < \frac{1}{k}$ then $k\varepsilon > 1$ Then $(k-2)n^{k\varepsilon-1} \to 0$ as $n \to \infty$.

$=> \lim_{n\to\infty} p\left(X \geq \frac{n}{2}\right) = 0$

$\exists n'$ s.t $\forall n > n'$ $p\left(X \geq \frac{n}{2}\right) < \frac{1}{2}$.

By <u>Lemma</u> Let k>0 be an integer and let p=p(n)≥ $(6k \ln n)n^{-1}$ for large n.

Then $\lim_{n\to\infty} p\left(\alpha \geq \frac{n}{2k}\right) = 0$

Proof <u>Lemma</u>: For $n \geq r > 2$.

$p(\alpha \geq r) \leq \binom{n}{r} 1 - p^{\binom{r}{2}} = \binom{n}{r}(1-p)^{\frac{r(r-1)}{2}} \leq n^r(1-p)^{\frac{r(r-1)}{2}}$

Since $e^{-p} \geq 1 - p$.

$(n(1-p)^{\frac{(r-1)}{2}})^r \le (n(e^{-p})^{\frac{r-1}{2}})^r = (ne^{\frac{-pr+p}{2}})^r$ Let p=$(6k\ln n)n^{-1}$,r=$\frac{n}{2k}$

=$\left(ne^{-\frac{3}{2}\ln n+\frac{p}{2}}\right)^r = \left(nn^{\frac{-3}{2}}e^{\frac{p}{2}}\right)^r = \left(n^{\frac{-1}{2}}e^{\frac{p}{2}}\right)^r$ Since $\left(n^{\frac{-1}{2}}e^{\frac{p}{2}}\right) \le 1$

$\le n^{\frac{-1}{2}}e^{\frac{p}{2}} \to 0$ as $n \to \infty$.

$\Rightarrow p\left(\alpha \ge \frac{n}{2k}\right) \to 0$ as $n \to \infty$.

Since $n^{\varepsilon-1} = n^{\varepsilon}n^{-1} \ge (6k\ln n)n^{-1}$ for large n,

Then $p\left(\alpha \ge \frac{n}{2k}\right) < \frac{1}{2}$ for large n.

Then exists a graph H, |H|=n

The number of cycles of length at most k at most $\frac{n}{2}$ and $\alpha(H) \le \frac{n}{2k}$.

Construct a new graph G

在 H 中每一個長度不大於 k 的 cycle delete 一個點

Then g(G)>k and $|G| \ge \frac{n}{2}$.

$\chi(G) \ge \frac{|G|}{\alpha(G)} \ge \frac{\frac{n}{2}}{\alpha(H)} > \frac{n/2}{n/2k} = k$

Then $\chi(G)g(G) > kk = k^2$.

8. $Let\ \aleph(G) = r+1$, $then\ H\ not\ in\ T_r(n)$ $(If\ H \le T_r(n),\ then\ \aleph(G) \le r)$

$\Rightarrow ex(n\ ;\ H) \ge \|T_r(n)\|.$

$Since\ H \le K_{(r+1)(t)}\ for\ some\ t\ ,\quad \|T_r(n)\| \le ex(n\ ;\ H) \le \|T_r(n)\| + \varepsilon n^2$

$(By\ Erdos\ and\ stone's\ Theorem\ \ ex(n\ ;\ H) \le \|T_r(n)\| + \varepsilon n^2\ )$

$$\frac{\|T_r(n)\|}{\binom{n}{2}} \le \frac{ex(n\ ;\ H)}{\binom{n}{2}} \le \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{\varepsilon n^2}{\binom{n}{2}}$$

$$\le \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{2\varepsilon n^2}{n(n-1)}$$

$$= \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{2\varepsilon}{1-\frac{1}{n}} \qquad \left(\ \|H\| \ge 1 \Rightarrow |H| = n \ge 2 \Rightarrow \frac{1}{1-\frac{1}{n}} \le 2\ \right)$$

$$\le \frac{\|T_r(n)\|}{\binom{n}{2}} + 4\varepsilon$$

$$Claim\ \lim_{n\to\infty} \frac{\|T_r(n)\|}{\binom{n}{2}} = 1-\frac{1}{r}$$

$$Let\ n = qr\ \ ,\ \|T_r(n)\| = \binom{n}{2} - r\binom{q}{2} = \binom{n}{2} - \frac{rq(q-1)}{2}$$

$$= \binom{n}{2} - \frac{rq(qr-r)}{2r} = \binom{n}{2} - \frac{n(n-r)}{2r}$$

$$= \binom{n}{2}(1-\frac{1}{r}) + \frac{n(r-1)}{2r}$$

$$\lim_{n\to\infty} \frac{\|T_r(G)\|}{\binom{n}{2}} = \lim_{n\to\infty}(1-r) + \frac{(r-1)n}{2r} \times \frac{2}{n(n-1)} = 1-\frac{1}{r}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\|T_r(G)\|}{\binom{n}{2}} = 1-\frac{1}{r} = \frac{r-1}{r} = \frac{\aleph(G)-2}{\aleph(G)-1}$$

李光祥

9. Prove that the diameter of a connected graph G is less than the number of distinct eigenvalues of G.

pf：

Let A be the adjacency matrix and r be the of distinct eigenvalues.

Let them be $\lambda_1,\lambda_2,\ldots,\lambda_r$.

Then $m(x)=\prod_{i=1}^{r}(x-\lambda_i)$ is the minimal polynomial of A, that is, m(A)=0.

This implies that some linear combinations of I, A, A^2, …, A^r are 0, and I, A, A^2, …, A^t are linearly independent for $t < r$.

Now we need to show that I, A, A^2, …, A^k are linearly independent when $k \le \text{diam}(G)$.

It suffices to show that A^k is not a linear combinations of I, A, A^2, …, A^{k-1} for all $k \le \text{diam}(G)$.

Choose $v_i, v_j \in V(G)$ such that $d(v_i, v_j) = k$.

Then this implies that $A^k(i,j) > 0$ and $A^t(i,j) = 0$ for $k \le \text{diam}(G)$.

Therefore, A^k is not a linear combinations of I, A, A^2, …, A^{k-1} for all $k \le \text{diam}(G)$.

Thus I, A, A^2, …, A^k are linearly independent when $k \le \text{diam}(G)$.

Hence $\text{diam}(G) < r$. □

劉宜君

#10 Prove that if G is a graph in each any two distant vertices have exactly one common neighbor, that $\Delta(G)=|G|-1.$

Pf. Case 1: G is regular.

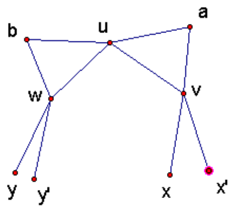
Since G is regular, and each any two distant vertices in G have exactly one common neighbor , then it is easy to see that $\lambda=\mu=1.$

By theorem : If G is strongly regular with n vertices and parameters $k,\lambda,\mu,$ then the two numbers below are nonnegative integers $\frac{1}{2}\left(n-1\pm\frac{(n-1)(\mu-\lambda)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right).$

We know that $\frac{1}{2}\left(n-1\pm\frac{k}{\sqrt{(k-1)}}\right)\in\square \;\Rightarrow\; \frac{k}{\sqrt{(k-1)}}\in\square \;\Rightarrow k=2.$

$\Rightarrow K_3$ is the only 2-regular graph satisfying the condition, and $\Delta(K_3)=|K_3|-1.$

Case 2 : G is not regular.



$\forall w,v\in V(G),\ |N(w)\cap N(v)|=1 \quad\Rightarrow$ forbids C_4.

If w does not adject to v in G ,let $\{u\}=N(w)\cap N(v).$

And let $\{a\}=N(u)\cap N(v),\ \{b\}=N(w)\cap N(u).$

$\forall x\in S,$ where S is $N(v)$ - $\{u,a\}$ has common neighbor $f(x)$ with $w.$

If $f(x)=b,\ \{b,u,v,x\}$ obtain $C_4.$

If $f(x)=w,\ \{w,x,v,u\}$ obtain $C_4.$

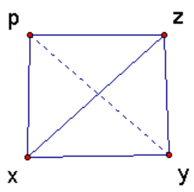
If $f(x)=f(x'),\ \{x',v,x,f(x)\}$ obtain $C_4,\ x'\in S\ ,x'\neq x.$

$\Rightarrow d(w)\geq d(v)$.

Similarly condition $\Rightarrow d(v)\geq d(w)$.

$\Rightarrow d(w)=d(v)$

Since G is not regular, $\exists x,y\in V(G),$ s.t. $d(x)\neq d(y).$



If $d(x)\neq d(y) \Rightarrow_{by\ above} x\sim_G y$, and let $\{z\}=N(x)\cap N(y).$

Since $d(z)$ can't equal to $d(x)$ and $d(y)$ at the same time ($d(x)\neq d(y)$), W.L.O.G. , let $d(z)\neq d(y)$.

If $\exists p\in V(G),\ p\notin N(y)$, then $d(p)=d(y)\neq d(x)$,and $d(p)=d(y)\neq d(z)\Rightarrow p\sim_G x$ and $p\sim_G z$

$\Rightarrow$ obtain C_4 , $\{p,x,y,z\}$ -><-

$\Rightarrow p(y)=n-1.$

By case1 and case2 ,we get $\Delta(G)=|G|-1.$