劉士慶

1. Prove that ex (n; K_{r+1}) $=\|T_r(n)\|$) and $T_r(n)$ is the unique extremal graph.

Proof: Since $T_r(n)$ does not contain a subgraph $K_{r+1}, ex(n;K_{r+1}) \geq \|T_r(n)\|$.

Claim: $K_{r+1}, ex(n;K_{r+1}) \leq \|T_r(n)\|$.

Method1:

Lemma: Let G be an r-partite graph of order n. then $\|G\| \leq \|T_r(n)\|$.

By lemma, it suffices to show that if G does not contain a subgraph K_{r+1},

then there exist an r-partite graph H s.t. $\|G\| \leq \|H\|$.

By induction on r.

It is clear that r=1 is true.

Let G be a graph with $\square(G)$ and G does not a subgraph K_{r+1}.

Let $x \in V(G)$ s.t. $\deg_G(x) = \Delta(G)$ and $G^{'} =< N_G(x) >_G$.

Clearly, G' does not contain K_r as a subgraph(如果有的話,G 就包含 K_{r+1}).

By induction hypothesis, there exists an (r-1)-partite H' s.t. $\|G'\| \leq \|H'\|$.

Let S=$V(G)\setminus V(H')$. Then the graph H' $\vee$ S is an r-partite graph with

$\|H'\| + [n-\Delta(G)]\square\Delta(G)$ edges.

Hence $\|T_n(n)\| = \|H'\| + [n-\Delta(G)]\square\Delta(G) > \|G'\| + \sum_{v\in S}\deg_G(v) \geq \|G\|$.

Proof 2: Let G be the extremal graph of order n. $\Rightarrow$ ||G||=ex (n, K_{r+1}).

Induction on n.

<i> n$\leq$r. It is clear that G is K_r and $T_r(n) = \|K_r\| = \|G\|$.

<ii> n>r. Claim: G does not contain $K_{r+1} \Rightarrow$ G contain K_r.

Proof of claim: If making x and y adjacent to create K_{r+1}, the other r-1 vertices must

be all adjacent to x and y. Thus x or y forms a K_r with the r-1 vertices.

$\because$ G does not contain $K_{r+1} \Rightarrow$ G contain K_r.

Let V (K_r) =W and V (G)\W=U, G' be the graph of U.

$\because$ G does not contain K_{r+1}.

$\therefore$Every vertex not in W has at most r-1 neighbor in W.

$\Rightarrow \| E(G) \| \le \binom{r}{2} + (r-1)(n-r) + \| < U >_G \|$

$\because$ G' does not contain $K_{r+1} \Rightarrow \| < U >_G \| \ \le \ \| T_r(n-r) \|$.

Then $\| E(G) \| \le \binom{r}{2} + (r-1)(n-r) + \| T_r(n-r) \| = T_r(n)$.

Proof of uniqueness: Let ||G||= $\| T_r(n) \|$ and G be an extremal graph.

Let y$\in$V (G), $\deg_G(y) = \delta(G)$.

Then ||G-y||= ||G||-. (By induction).

G-y must be isomorphic to $T_r(n-1)$.

$\Rightarrow$Then the smallest partite set in G-y has $\left\lfloor \frac{n-1}{r} \right\rfloor$vertices.

Hence $|<y, G-y>|$= (n-1)- $\left\lfloor \frac{n-1}{r} \right\rfloor$ = n-$\left\lfloor \frac{n}{r} \right\rfloor$.

$\because$ G-y is $T_r(n-1)$.

$\therefore$ y 最少跟一個 partite set is non-adjacent.(否則會有 K_{r+1})

And the partite set must be of size$\left\lfloor \frac{n-1}{r} \right\rfloor$. Hence G$\cong T_r(n)$.

李光祥

2. Prove that $ex(n;K_{s,t}) \le \frac{1}{2}(s-1)^{\frac{1}{t}}n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$, t ≤ s.

(Give at least two different proofs.)

pf:

(First proof)

Let G be a graph of order n without $K_{s,t}$ with a maximal number of edges.

Let the degree sequence of G be <$d_1, d_2, \ldots, d_n$>, $d_1 \le d_2 \le \ldots \le d_n$.

Consider the t-stars in G.

Since G does not contain $K_{s,t}$, there are at most $\binom{n}{t}(s-1)$ t-stars in G.

Let the average degree of G be $d = \frac{2\|G\|}{n}$.

Then we have $\binom{n}{t}(s-1) \ge \sum_{i=1}^{n}\binom{d_i}{t} \ge n\binom{d}{t}$.

$$\Rightarrow (s-1)n^{-1} \ge \frac{\binom{d}{t}}{\binom{n}{t}} = \frac{d(d-1)\ldots(d-t+1)}{n(n-1)\ldots(n-t+1)} \ge \left(\frac{d-t+1}{n}\right)^t.$$

$$\Rightarrow (s-1)^{\frac{1}{t}}n^{-\frac{1}{t}} \ge \frac{d-t+1}{n}.$$

$$\Rightarrow \frac{2\|G\|}{n} = d \le (s-1)^{\frac{1}{t}}n^{1-\frac{1}{t}} + (t-1).$$

$$\Rightarrow \|G\| \le \frac{1}{2}(s-1)^{\frac{1}{t}}n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n.$$

(Second proof)

Lemma :

Let 2 ≤ s ≤ m, 2 ≤ t ≤ n, 0 ≤ r < m, z = km+r and z = my.

Let $G_2(m,n)$ be a graph of size z that does not contain $K_{s,t}$.

Then $m\binom{y}{t} \le (m-r)\binom{k}{t} + r\binom{k+1}{t} \le (s-1)\binom{n}{t}$.

pf:

Let $G_2(m,n) =$ (V_1,V_2), where |V_1| = m and |V_2| = n.

Define $H = \left(V_1, \binom{V_2}{t}\right)$, where $\binom{V_2}{t}$ is the collection og all t-subsets of V_2.

$\forall x \in V_{1,} A \in \binom{V_2}{t}, x \square A$ iff $x \square a, \forall a \in A$.

$$\Rightarrow \|H\| = \sum_{x \in V_1} \binom{\deg_G(x)}{t}.$$

Note that $\forall B \in \binom{V_2}{t}, \deg_G(B) \le s-1$.(Otherwise $K_{s,t} \le G$.)

$$\Rightarrow \|H\| = \sum_{x \in V_1} \binom{\deg_G(x)}{t} \le (s-1)\binom{n}{t}.$$

Since $\sum_{x \in V_1} \deg_G(x) = \|G\| = z = km + r = my$, we have

$$m\binom{y}{t} \le (m-r)\binom{k}{t} + r\binom{k+1}{t} \le \sum_{x \in V_1} \binom{\deg_G(x)}{t} \le (s-1)\binom{n}{t}.$$

Theorem :

$$z(m,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m.$$

pf:

By the above lemma, we have $m\binom{y}{t} \le (s-1)\binom{n}{t}$.

$$\Rightarrow \frac{\binom{y}{t}}{\binom{n}{t}} \le (s-1)m^{-1}.$$

$$\Rightarrow \frac{y(y-1)\dots(y-t+1)}{n(n-1)\dots(n-t+1)} \le (s-1)m^{-1}.$$

Since $G \le K_{m,n}, my = z = \|G\| \le \|K_{m,n}\| = mn$ and $y \le n$.

$$\Rightarrow \frac{(y-t+1)(y-t+1)\dots(y-t+1)}{(n-t+1)(n-t+1)\dots(n-t+1)} \le \frac{y(y-1)\dots(y-t+1)}{n(n-1)\dots(n-t+1)} \le (s-1)m^{-1}.$$

$$\Rightarrow \frac{(y-t+1)^t}{(n-t+1)^t} \le (s-1)m^{-1}.$$

$$\Rightarrow (y-t+1)^t \le (s-1)(n-t+1)^t m^{-1}.$$

$$\Rightarrow y-t+1 \le (s-1)^{\frac{1}{t}}(n-t+1)m^{-\frac{1}{t}}.$$

$$\Rightarrow \frac{z}{m} = y \le (s-1)^{\frac{1}{t}}(n-t+1)m^{-\frac{1}{t}} + (t-1).$$

$$\Rightarrow z \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m.$$

Now we prove this exercise.

By the above theorem, we have

$$z(m,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m .$$

$\Rightarrow z(n,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)n^{1-\frac{1}{t}} + (t-1)n \le (s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + (t-1)n .$

Hence it suffices to claim $ex(n;K_{s,t}) \le \frac{1}{2} z(n,n;s,t)$.

Let G be a graph without $K_{s,t}$ with a maximal number of edges and

$V(G) = \{x_1, x_2, ..., x_n\}$.

Define H = (V_1,V_2) such that $V_1 = \{x_1', x_2', ..., x_n'\}, V_2 = \{x_1'', x_2'', ..., x_n''\}$ and

$x_i' \square x_j''$ iff $x_i \square x_j$.

Since G does not contain $K_{s,t}$, we have H does not contain $K_{s,t}$, and hence

$\|H\| \le z(n,n;s,t)$.

By the construction of H, we have $\|H\| = 2\|G\|$.

$\Rightarrow ex(n;K_{s,t}) = \|G\| \le \frac{1}{2}\|H\| \le \frac{1}{2} z(n,n;s,t) \le \frac{1}{2}(s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n .$ □

3. Find a graph G of order 100 s.t. G does not contain $K_{2,2}$ and $\| G \| \approx 500$

Proof. Since $ex(n; K_{2,2}) \leq \frac{n}{4}(1 + \sqrt{4n-3})$ and equality holds whenever $n = q^2 + q + 1$, q is a prime power. Let $q = 3^2$, then

$$n = 91 \quad \text{and} \quad ex(91; K_{2,2}) = \frac{91}{4}(1 + \sqrt{4 \cdot 91 - 3}) = 455$$

Therefore exist a graph G' does not contain$K_{2,2}$ with $\| G' \| = 455$ and $|G'| = 91$. Let $G = G' \cup C_9$. Since G' and C_9 does not contain $K_{2,2}$, therefore

$$G \text{ does not contain } K_{2,2} \quad \text{and} \quad |G| = 100, \| G \| = 455 + 9 = 464$$

□

陳巧玲

4. Find a C_4-saturated graph of order n with minimum number of edges for each positive integer n.

First, we construct C_4 − saturated *graph* with $\left\lfloor \frac{3n-5}{2} \right\rfloor$ *edges*.

Second, we prove such graph with minimum number of edges for each integer n.

Now, we construct C_4-saturated graph with order n.

Step 1:

Take $t=\left\lfloor \frac{n-3}{2} \right\rfloor$ triangle sharing a vertex w.

Step2.

Join $p=n-2t-1=n-2\left\lfloor \frac{n-3}{2} \right\rfloor-1=\begin{cases} 3, n=2k+4 & k\in\{0\}\cup Z^{+} \\ 2, n=2k+3 & k\in\{0\}\cup Z^{+} \end{cases}$

new vertices to p vertices of one of these triangles using indepedednt edges,(i.e the new vertices have no common neighbors.)

So,we want to prove the graph we obtained is C_4 − *saturated*.

(*case*1)

(*a*)If add $\{p_1, p_2\}$ to be new edge:

Then, clearly $p_1-a_1-a_2-p_2-p_1$ is C_4 ,where a_1, a_2 are vertices of one of the t triangles.

(b)Otherwise, add any edge $\{x,z\}$ $\{x,z\}\neq\{p_1, p_2\}$:

Then, for $x\in V(G)$, $|N(x)\cap N(y)|=1$.

Moreover, $\{z, w\}\in E(G)$.

So, let $y\in N(x)\cap N(w)$

Then x-z-w-y-x is C_4.

Hence, by (a) (b), when n=2k+3, graph is C_4 − *saturated*.

($case$ 2)

n=2k+4 ,k $\in \{0\} \cup Z^{+}$.

Let the new three vertices p_1, p_2, p_3, where $|N(p_3) \cap w| = 0$.

Consider two cases:

(a) If add $\{x, z\}$ to be new edge ,x,z $\neq p_3$.

Then ,we can use the same way of case1 to discuss.

(b) If x or z is p_3, then w.l.o.g let x= p_3.

Then, except p_3,

$\forall x_1 \in V(G), |N(x_1) \cap N(w)| = 1$

Thus, $|N(z) \cap N(w)| = 1$.

Let y $\in N(z) \cap N(w)$

Then y-z-x-w-y is C_4.

Hence, by (a) (b) ,when n=2k+4, graph G is $C_4 - saturated$.

Hence, by case1 and case2, the graph we constructed is $C_4 - saturated$.

Moreover, the edges of graph are $\left\lfloor \frac{3n-5}{2} \right\rfloor$.

Then, after we constructing C_4 − saturated graph, let's

prove its size is minimum.

Let P_k be a path on k edges.

A graph is $P_k -$connected if all pairs of non-adjacent vertices are $P_k -$connected.

So, if adding an edge between two vertices creates a C_k,then a P_{k-1} must connect the two vertices.

Thus, a C_k-saturated graph is $P_{k-1} -$connected.

Now, we want to prove the following results:

Let G be a $P_3 - connected$ graph, then $\|G\| \geq \frac{3(n-2)}{2}$.

(Lemma)

Let G be a connected graph. If each edge of G is in C_3, then $\|G\| \geq \frac{3(n-1)}{2}$.

(pf)

Since G is connected , G has a spanning tree T, $\|T\| \geq n-1$.

An edge of G-T can form a C_3 with at most two edges of T.

So, $\|G\text{-}T\| \geq \frac{1}{2}(n\text{-}1)$.

Thus, $\|T\| \geq (n-1) + \frac{1}{2}(n\text{-}1) = \frac{3}{2}(n\text{-}1)$.

(Thm)

Let G be a $P_3 -$connected graph . Then $\|G\| \geq \frac{3}{2}(n\text{-}2)$.

(pf)

It is true for n=1.

So, assume $n \geq 2$.

Since G is connected, $\delta>0$.

If $\delta \geq 3$, then $\|G\| \geq \frac{3}{2}n > \frac{3}{2}(n\text{-}2)$.

So, assume $\delta=1$ or $\delta = 2$.

(case1)

δ=1

Let deg(v)=1 and N(v)=$\{w\}$.

Let T be the breath-first tree of G rooted at v.

Let X=$\{x|d(x,v)=2\}$ Y=$\{x|d(x,v)=3\}$

Since G is P_3- connected , all vertices are distance at most 3 from v.

So, all vertices other than v and w are in X or Y.

Thus, $|X|+|Y|=n-2$.

Because x $\in$ X is not adjacent to v, there is a path v-w-q-x.

Since d(q,x)=2, q $\in$ X.

Hence, q $\in$ X, N(q) $\cap$ X $\neq \phi$.

Let $Y_1=\{z|\deg(z)=1, z\in V(Y)\}, |Y_1|=k$

Let $Y_2=Y-Y_1$.

Let X_1 be the vertices of X adjacent to Y_1, and let X_2=X-X_1

Then, for all y_1 $y_2 \in Y_1$, since G is P_3 – connected, there is a path $y_1-x_1-x_2-y_2$,

where $x_1, x_2 \in X_1$.

Moreover, each vertex in X_1, is adjacent to exactly one vertex in Y_1, and every pair of vertces in X_1 are adjacent.

Thus, $|X_1|=|Y_1|=k$, and X_1 is clique.

Since each vertices in X_2 and Y_2 is incident to at least one edge not in T, the number of edges noy in T

is $\|G-T\|\geq\binom{|X_1|}{2}+\frac{|X_1|+|X_2|}{2}=\binom{k}{2}+\frac{n-2-2k}{2}\geq\frac{n-4}{2}$.

$\rightarrow\|G\|\geq\frac{1}{2}(n-4)+n-1=\frac{3}{2}(n-2)$.

(case2)

δ=2 and $\exists$v is not in C_3, where deg(v)=2.

Let T be the breath-first tree of G rooted at v.

Let X=$\{x|d(x,v)=2\}$ Y=$\{x|d(x,v)=3\}$

Since a P_3 connects v to x, x $\in X_1$

There is an edge incident to x whiach is not in T.

Further, since δ=2, each y $\in$ Y is also incident to an edge not in T.

So,$\|G-T\|\geq\frac{1}{2}(|X|+|Y|)=\frac{1}{2}(n-3)$.

Therefore, $\|G\|\geq\frac{1}{2}(3n-5)$.

(case3)

δ=2 and $\forall$v is in C_3, wherw deg(v)=2.

Let S be the subgraph of all C_3's with one or more degree 2 vertices.

Let S_i $1 \le i \le m$ be the componets of S.

By Lemma 1, $\|S_i\| \ge \frac{3}{2}(|S_i|-1)\ \forall i.$

Moreover, since the degree 2 vertices of two components are P_3 –connected ,there is an edge between every pair of components of S.

So,there are at least $\binom{m}{2}$ edges between components of S.

Then, the number of edges inthe induced subgraph induced by the vertices of S is $\|<S>\| \ge \sum_{i=1}^{m}\|S_i\| + \binom{m}{2} \ge$

$$\sum_{i=1}^{m}\frac{3}{2}(|S_i|-1)+\binom{m}{2} = \frac{3}{2}|S| + \frac{m^2}{2} - 2m \ge \frac{3}{2}|S| - 2.$$

Since all degree 2 vertices are in S,verticess not is S have 3 or more.

Thus, $\|G\| \ge \frac{3}{2}|S| - 2 + \frac{3}{2}(|G|-|S|) = \frac{3}{2}n - 2.$

Hence, by case1,case2,case3, we can conclued $\|G\| \ge \frac{3}{2}(n-2).$

Thus, $\|G\| \ge \left\lceil \frac{3n-6}{2} \right\rceil = \left\lfloor \frac{3n-5}{2} \right\rfloor.$

So, the graph we construct is the graph of minimum size.

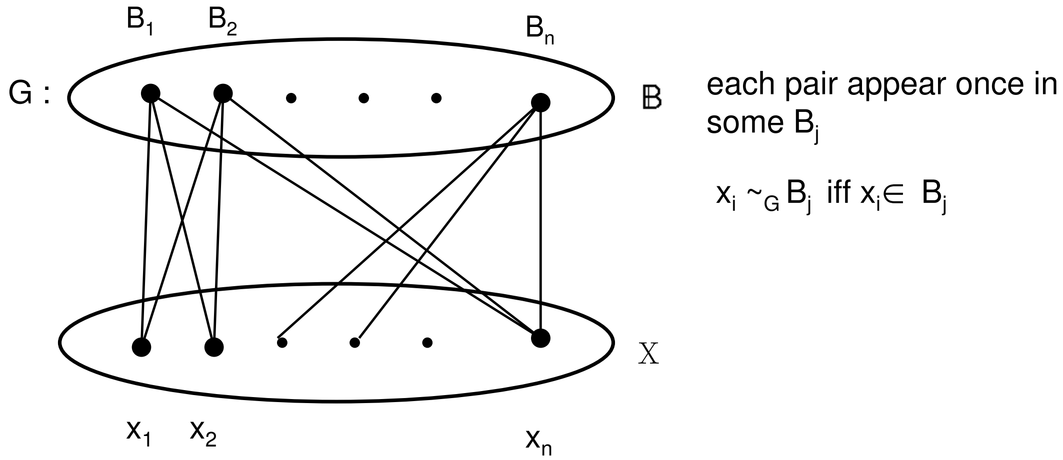
Graph Theory

Hui-Fen,Tseng

6. Find $z(n, n; 2, 2)$ for as many n as possible.
sol:

When $n = q^2 + q + 1$ which q is prime power, $z(n, n; 2, 2) \leq \frac{n}{2}[1 + (4n - 3)^{\frac{1}{2}}]$ holds.

Since q is prime power, there exists a projective plane, let $G = (\mathbf{X}, \mathbb{B})$ be $||G|| = (q + 1)(q^2 + q + 1)$. And since G is projective plane, it doesn't contain $K_{2,2}$. Since $C_4 \cong K_{2,2}$, and G is $2-(q^2+q+1, q+1, 1)$ design.



Since $n = q^2 + q + 1$, it implies $q^2 + q + (1 - n) = 0$, we obtain $q = \frac{-1 \pm \sqrt{1-4(1-n)}}{2} = \frac{-1 \pm (4n-3)^{\frac{1}{2}}}{2}$ (we choose postivie). Thus, $||G|| = (q+1)(q^2+q+1) = \frac{1+(4n-3)^{\frac{1}{2}}}{2} \cdot n$. We obtain $z(n, n; s, t) \leq \frac{n}{2}[1+(4n-3)^{\frac{1}{2}}]$ as required.

7. Find a subgraph G of $K_{9,12}$ which is C_4-saturated and $\|G\| = 28$ (Explain your answer).

Definition: (1). A partial $2-(v,K,1)$ design $(\mathbb{X},\mathbb{B})$ is said to be an extension of a $2-(v,K,1)$ design $(\mathbb{X},\mathbb{B}')$ if $\mathbb{B} \neq \mathbb{B}'$, $|\mathbb{B}| = |\mathbb{B}'|$ and $B' \subseteq B$ for each $B \in \mathbb{B}$ and $B' \in \mathbb{B}'$.
(2). A partial $2-(v,K,1)$ design $(\mathbb{X},\mathbb{B})$ is asid to be non-extebtable if it has no extension.

Lemma 1 The variety-block graph $G_{(\mathbb{X},\mathbb{B})}$ of a partial $2-(v,K,1)$ design $2-(v,K,1)$ is a C_4-saturated graph of $K_{|\mathbb{X}|,|\mathbb{B}|} \Leftrightarrow (\mathbb{X},\mathbb{B})$ is non-extentable.

Proof of Lemma 1:
($\Rightarrow$) Let the variety-block graph $G_{(\mathbb{X},\mathbb{B})}$ of a partial $2-(v,K,1)$ design $(\mathbb{X},\mathbb{B})$ be a C_4-saturated graph of $K_{|\mathbb{X}|,|\mathbb{B}|}$. If $(\mathbb{X},\mathbb{B})$ is extendable then let $(\mathbb{X},\mathbb{B}')$ be the extension of $(\mathbb{X},\mathbb{B})$ and $x \in B' \backslash B$ where $B' \subseteq B$, $B \in \mathbb{B}$ and $B' \in \mathbb{B}'$. Since $(\mathbb{X},\mathbb{B}')$ is also a partial $2-(v,K,1)$ design,adding the edge xB to $G_{(\mathbb{X},\mathbb{B})}$ won't make C_4. Hence $G_{(\mathbb{X},\mathbb{B})}$ is not a C_4-saturated ($\rightarrow\leftarrow$). Thus $(\mathbb{X},\mathbb{B})$ is non-extendable.
($\Leftarrow$) Suppose $(\mathbb{X},\mathbb{B})$ is non-extendable. If $G_{(\mathbb{X},\mathbb{B})}$ is not C_4-saturated, then there exists an edge such that adding the edge to $G_{(\mathbb{X},\mathbb{B})}$ is also C_4-free. W.L.O.G. we let this edge be xB. Then let $\mathbb{B}'$ be the set of blocks obtained from $\mathbb{B}$ by replacing B with $B' = B \cup \{x\}$. Hence $(\mathbb{X},\mathbb{B}')$ is an extension of $(\mathbb{X},\mathbb{B})$ ($\rightarrow\leftarrow$).
(Since adding xB to $G_{(\mathbb{X},\mathbb{B})}$ is also C_4-free, $(\mathbb{X},\mathbb{B}')$ is also a partial $2-(v,K,1)$ design. Moreover, $\mathbb{B} \neq \mathbb{B}'$, $|\mathbb{B}| = |\mathbb{B}'|$ and $B' \subseteq B$ for each $B \in \mathbb{B}$ and $B' \in \mathbb{B}'$. Hence $(\mathbb{X},\mathbb{B}')$ is an extension of $(\mathbb{X},\mathbb{B})$). $\square$

sol: Let $\mathbb{X} = \{1,2,3,4,5,6,7,8,9\}$

$$\begin{aligned}\mathbb{B} = \{&\{1,2,3,4,5,6,7\},\{1,8,9\},\{2,9\},\{3,9\},\{4,9\},\{5,9\},\\ &\{6,9\},\{7,9\},\{2,8\},\{3,8\},\{1\},\{1\}\} \Rightarrow |\mathbb{B}| = 12\end{aligned}$$

It's easy to see that $(\mathbb{X},\mathbb{B})$ is a partial $2-(9,K,1)$ design, where

$K = \{1, 2, 3, 7\}$. (Since except $(4, 8), (5, 8), (6, 8), (7, 8)$ these four pairs, each pair of $\mathbb{X}$ occurs in exactly one bolck).

Claim: $(\mathbb{X}, \mathbb{B})$ **is non-extendable**.

It sufficies to prove that if we add x ($x \in \mathbb{X}$) to any block of $\mathbb{B}$, then there exists a pair which occurs in at least two blocks.

(1). If we add 8 or 9 to $\{1, 2, 3, 4, 5, 6, 7\}$, then $(i, 8)$ or $(i, 9)$ for some $i = 1, 2, ..., 7$ will occurs in at least two blocks.

(2). If we add j to $\{1, 8, 9\}$, $j = 2, 3, ..., 7$, then $(1, j)$ will occurs in two blocks.

(3). If we add j to $\{i, 9\}$, $i = 2, 3, ..., 7$, $j \in \mathbb{X} \backslash \{i, 9\}$,

$$\begin{cases} \text{if } j = 8, \text{then } (8,9) \text{occurs in two blocks;} \\ \text{if } j \neq 8, \text{then } (i,j) \text{occurs in two blocks.} \end{cases}$$

(4). If we add j to $\{i, 8\}$, $i = 2, 3$, $j \in \mathbb{X} \backslash \{i, 8\}$, then (i, j) occurs in at least two blocks.

(5). If we add j to $\{1\}$, $j = 2, 3, ..., 9$, then $(1, j)$ occurs in two blocks.

$\Rightarrow$ by (1) (5) we have $(\mathbb{X}, \mathbb{B})$ is non-extendable.

Hence by Lemma 1 we have, $G_{(\mathbb{X},\mathbb{B})}$ is a C_4-saturated graph of $K_{9,12}$ and $\|G_{(\mathbb{X},\mathbb{B})}\| = \sum_{B \in \mathbb{B}} |B| = 7 + 3 + 2 \times 8 + 1 \times 2 = 28$. □

劉宜君

8. *Let* $\aleph(G) = r+1$, *then* H *not in* $T_r(n)$ (*If* $H \le T_r(n)$, *then* $\aleph(G) \le r$)

$\Rightarrow ex(n\ ;\ H) \ge \|T_r(n)\|$.

Since $H \le K_{(r+1)(t)}$ *for some* t , $\|T_r(n)\| \le ex(n\ ;\ H) \le \|T_r(n)\| + \varepsilon n^2$

(*By Erdos and stone's Theorem* $ex(n\ ;\ H) \le \|T_r(n)\| + \varepsilon n^2$)

$$\frac{\|T_r(n)\|}{\binom{n}{2}} \le \frac{ex(n\ ;\ H)}{\binom{n}{2}} \le \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{\varepsilon n^2}{\binom{n}{2}}$$

$$\le \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{2\varepsilon n^2}{n(n-1)}$$

$$= \frac{\|T_r(n)\|}{\binom{n}{2}} + \frac{2\varepsilon}{1-\frac{1}{n}} \quad \left(\|H\| \ge 1 \Rightarrow |H| = n \ge 2 \Rightarrow \frac{1}{1-\frac{1}{n}} \le 2 \right)$$

$$\le \frac{\|T_r(n)\|}{\binom{n}{2}} + 4\varepsilon$$

Claim $\lim_{n\to\infty} \frac{\|T_r(n)\|}{\binom{n}{2}} = 1 - \frac{1}{r}$

Let $n = qr$, $\|T_r(n)\| = \binom{n}{2} - r\binom{q}{2} = \binom{n}{2} - \frac{rq(q-1)}{2}$

$$= \binom{n}{2} - \frac{rq(qr-r)}{2r} = \binom{n}{2} - r\frac{n(n-r)}{2r}$$

$$= \binom{n}{2}(1-\frac{1}{r}) + \frac{n(r-1)}{2r}$$

$$\lim_{n\to\infty} \frac{\|T_r(G)\|}{\binom{n}{2}} = \lim_{n\to\infty}(1-r) + \frac{(r-1)n}{2r} \times \frac{2}{n(n-1)} = 1 - \frac{1}{r}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\|T_r(G)\|}{\binom{n}{2}} = 1 - \frac{1}{r} = \frac{r-1}{r} = \frac{\aleph(G)-2}{\aleph(G)-1}$$

林逸軒

9. Given n point in the plane.　Prove that there are most $\frac{1}{\sqrt{2}}n^{\frac{3}{2}}+\frac{n}{4}$ pairs with distance 1.

Proof:把 n 個 points 放在 plane 上

若 2 points x ,y 距離爲 1.則在 G 中有 xy 邊.

Goal:$|E(G)| \leq \frac{1}{\sqrt{2}}n^{\frac{3}{2}}+\frac{n}{4}$

Claim①:$2\binom{n}{2} \geq \sum_{v\in V(G)}\binom{\deg(v)}{2}$

左式爲計算 n 點中任兩點的 common neighbor 數.

Since 在平面中任兩點以距離一爲半徑作圓,

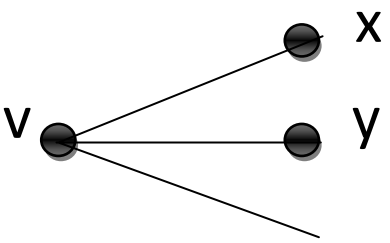
最多交兩點(相交圓).

意同於,任兩點最多有 2 個 common neighbor

所以最多 $2\binom{n}{2}$ 個.

右式爲從圖型上看 common neighbor 數

對每一點的 degree 中任兩點就有 1 common neighbor



x,y 有 common neighbor v

x,z 有 common neighbor v

- z

實際上有$\sum_{v\in V(G)}\binom{\deg(v)}{2}$個

$$\Rightarrow 2\binom{n}{2} \geq \sum_{v\in V(G)}\binom{\deg(v)}{2}$$

Claim②:$\sum_{v\in V(G)}\binom{\deg(v)}{2} \geq n\binom{\frac{2e}{n}}{2}$

Since $\sum_{v\in V(G)}\deg(v) = 2e$ where $e = |E(G)|$

By $\binom{n}{k}$ is convex function

又右式比左式平均 (i.e degree 的落差可能很大)

$$\Rightarrow \sum_{v\in V(G)}\binom{\deg(v)}{2} \geq n\binom{\frac{2e}{n}}{2}$$

by①②=> $2\binom{n}{2} \geq \sum_{v\in V(G)}\binom{\deg(v)}{2} \geq n\binom{\frac{2e}{n}}{2}$

$$\Rightarrow 2\,\frac{n\,(n-1)}{2} \geq n\,\frac{\frac{2e}{n}\,(\frac{2e}{n}-1)}{2}$$

$$\Rightarrow n\,(n-1) \geq e(\frac{2e-n}{n})$$

$$\Rightarrow n^2 - n \geq \frac{2e^2-en}{n} \Rightarrow 2e^2 - en - (n^3 n^2) \leq 0$$

$$=> \quad e \le \frac{n+\sqrt{n^2+8(n^3-n^2)}}{2*2} = \frac{n}{4} + \frac{\sqrt{8n^3-7n^2}}{4}$$

Since

$$\frac{1}{\sqrt{2}} n^{\frac{3}{2}} - \frac{\sqrt{8n^3-7n^2}}{4} = \frac{1}{4\sqrt{2}} (4\sqrt{n^3} - \sqrt{16n^3 - 14n^2}) > 0$$

Then $\frac{1}{\sqrt{2}} n^{\frac{3}{2}} + \frac{n}{4} \ge \frac{n}{4} + \frac{\sqrt{8n^3-7n^2}}{4} \ge e = E(G)$

=> most $\frac{1}{\sqrt{2}} n^{\frac{3}{2}} + \frac{n}{4}$ pairs with distance 1.

9622534 葉彬

May 26, 2008

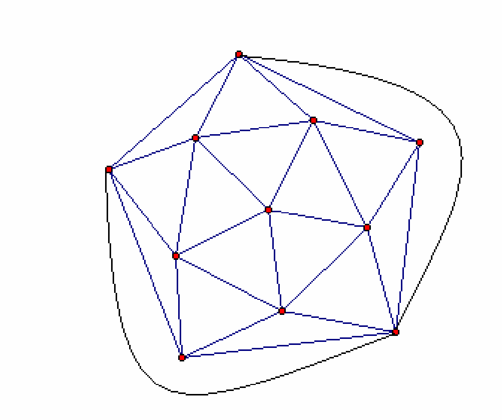
10. Construct a graph G of order 11 such that G does not contain $K_{3,3}$ and G has at least $\lfloor (11^{5/3})/2 \rfloor$ edges.

First of all, we calculate that $\lfloor (11^{5/3})/2 \rfloor = 27$.

(1)

If G is a planar graph, then $|E(G)| \le 3|V(G)| - 6 = 33 - 6 = 27$.

Thus there exists a planar graph with order 11 and size 27. Moreover, since this graph is planar, it contains no $K_{3,3}$.



This graph is a planar graph since there is no crossing.

It has 11 vertices and 27 edges.

(2)

Let $V(G) = K_1 \cup K_2 \cup K_3 \cup O$, where

$K_1 = \{K_{11}, K_{12}, K_{13}, K_{14}\}$

$K_2 = \{K_{21}\}$

$K_3 = \{K_{31}, K_{32}, K_{33}, K_{34}\}$

$O = \{O_1, O_2\}$, thus $|V(G)| = 11$.

Let $< K_1 \cup K_2 > \cong < K_2 \cup K_3 > \cong K_5$, complete graph of order 5,

$|E_{O,K_2}| = 0$, and O_1 is adjacent to O_2. Moreover, let

$O_1 \sim K_{11}, K_{12}, K_{31}, K_{32}$ and

$O_2 \sim K_{13}, K_{12}, K_{33}, K_{32}$.

$K_{1i} \sim K_{3i}, \forall i = 1, 2, 3, 4$.

Then the size of graph G would be

$$\| G \| = |< K_1 \bigcup K_2 >| + |< K_2 \bigcup K_3 >| + | E_{K_1,K_3} | + |< O >| + \sum_{i=1}^{2} \left(| E_{K_1,O_i} | + | E_{K_2,O_i} | + | E_{K_3,O_i} | \right)$$

$$= 10 + 10 + 4 + 1 + 2(2 + 0 + 2) = 33.$$

Claim: G is $K_{3,3}$-free.

Proof:

First $< K_1 \bigcup K_2 \bigcup K_3 >$ contains no $K_{3,3}$. If it does contain $G' \cong K_{3,3}$ as a subgraph, then there must exist $u, v \in V(G')$, $u \in K_1$ and $v \in K_3$ in G.

Then u and v have at least most 3 common neighbors, namely K_{21}, the only neighbor u have in K_3 and the only neighbor v have in K_1. But these 3 vertices have no more common neighbors but u and v, thus forms no $K_{3,3}$.

Then suppose $| V(G') \bigcap O | \neq \varnothing$.

(i) $| V(G') \bigcap O | = 1$

Then suppose it's O_1. There are 4 neighbors of O_1, besides O_2.

Any 3 of them have at most one common neighbor, namely K_{21}. Thus forms no $K_{3,3}$.

(ii) $| V(G') \bigcap O | = 2$

O_1 and O_2 can not be in the same partie in G', since they only have 2 common neighbors.

Suppose O_1 and O_2 are in different parties in G'. Then $K_{21} \notin V(G')$ since K_2 connects to neither O_1 nor O_2. In order to create a $K_{3,3}$, there must be 2 vertices that are in the same partie with O_1 in G' which are neighbors of O_1 or O_2 and have at least 2 common neighbors which are also neighbors of O_1 or O_2.

1) Suppose such 2 vertices, u and v, are in K_1.
   Then they only have one common neighbor fits conditions above.
2) Suppose u is in K_1, v is in K_3.
   If they are adjacent, then they have no common neighbor fits conditions above.
   If they are not adjacent, then they have 2 common neighbors, namely u's only neighbor in K_3, u', and v's only neighbor in K_1, v', which means if $u = K_{1i}$ for some i, $u' = K_{3i}$, similarly if $v = k_{3j}$, $v' = k_{1j}$ for some j.
   However, since O_1, u, v are in same partie in G', $O_1 \sim u'$ and $O_1 \sim v'$. Thus the nonordered pair (i, j) can only be (1, 2), which means O_2 doesn't connect to either u or v, since $O_2 \sim K_{1x}$ or K_{3x} if and only if x = 2 or 3. Thus this forms no $K_{3,3}$.

And we finish the proof for the claim.