Graph Theory II, Homework I　　李光祥

1. Prove that G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.

pf：

Claim：Deletion and contraction of edges preserve planarity.

Let G be a planar graph and given an embedding of G.

It is clear that deleting an edge can not introduce a crossing.

Let G^* be the dual graph of G.

Contracting an edge e in G has the effect of deleting the dual edge e^* in G^*.

Since deleting an edge can not introduce a crossing, $G^* - e^*$ is planar.

Since $G^* - e^*$ is planar and G/e is its planar dual, G/e is planar.

Claim：If G contains a subdivision of H, then H is a minor of G.

Since H can be obtained from G by deleting the edges not in the subdivision and contracting the edges incident to the vertices of degree 2, H is a minor of G.

(⇒) Since deletion and contraction of edges preserve planarity and K_5 and $K_{3,3}$ are nonplanar, these graphs can not be obtained from a planar graph G by deleting or contracting the edges, that is, neither K_5 nor $K_{3,3}$ is a minor of G.

(⇐) Suppose not. Then G is nonplanar.

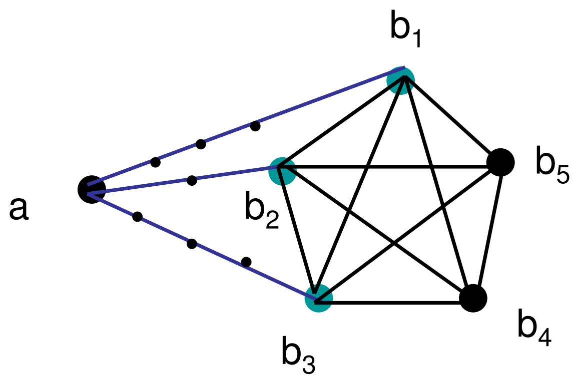
By Kuratowski's Theorem, G contains a subdivision of K_5 or $K_{3,3}$, and hence K_5 or $K_{3,3}$ is a minor of G, a contradiction. □

Graph Theory

Hui-Fen,Tseng

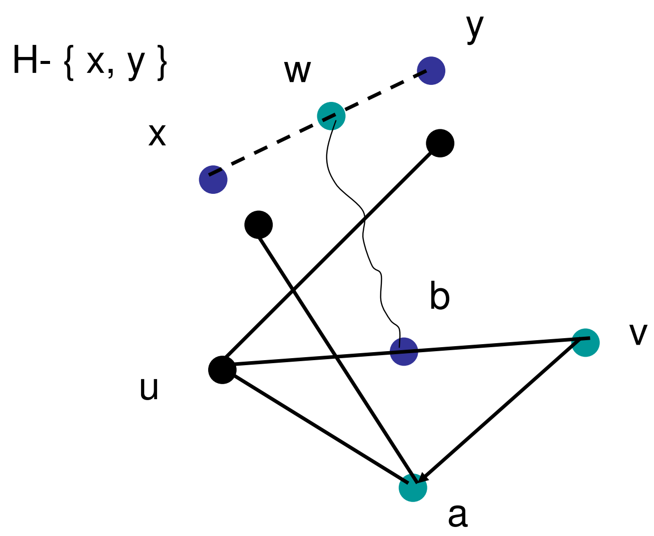
2. Prove that every 3-connected graph with at least 6 vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$.

Proof. Let G be a 3-connected graph with at least 6 vertices taht contain a subdivision of K_5, and H be a subdivision of K_5. We discuss two cases: $|H| = 5$ and $|H| > 5$.
(1) when $|H| = 5$.

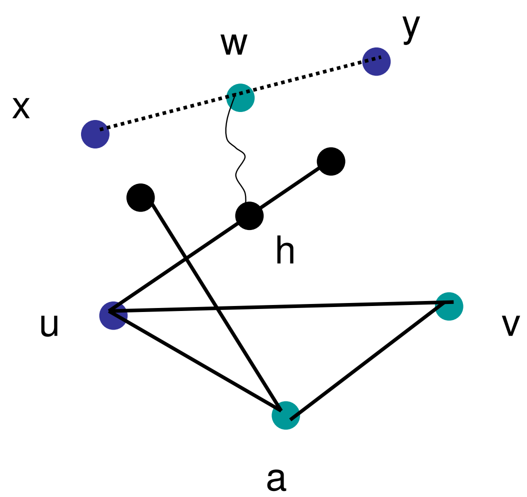


Since G is 3-connected and according to Menger's Theorem, there exists a vertex, denoted a, such that there are three vertice-disjoint paths from a to b_1, b_2, and b_3, respectively. Hence we have two partite sets, which are $\{a, b_4, b_5\}$ and $\{b_1, b_2, b_3\}$. Hence, G contains a subdivision of $K_{3,3}$. We hace done.
(2) when $|H| > 5$. Suppose x, y-path has at least one vertex where x, y-path on K_5. Since G is 3-connected, $G - \{x, y\}$ has the shortest path from $V(xy)$ to $V(H) - V(xy)$ where
$V(xy) = \{w | w$ is a vertex which belongs to $x, y-$path, but $w \notin x$ or $y\}$.
There exists a vertex in $V(xy)$, denoted w. Consider two cases.
Case 1:

If b is on cycle u, a, v, and not equal to v. Since $G - \{x, y\}$ has the shortest path from w to b. We can find two partite sets $\{x, y, b\}$, $\{w, v, a\}$



of G. Hence, G contains a subdivision of $K_{3,3}$.
Case 2:



If h is not on cycle u, v, a, W.L.O.G, let w, h-path be the shortest path from w to $V(H) - V(xy)$. We find two partite sets $\{x, y, u\}$ and $\{w, a, v\}$ of G. Why isn't h on set $\{x, y\}$? Since we aren't sure whether h is adjacent to a and v. Hence we pick u on the set $\{x, y\}$. Hence G contains a subdivision of $K_{3,3}$.

□

Graph Theory II, Homework I　　劉士慶

3. Estimate the thickness of $K_{m(n)}$ by given an upper bound and a lower bound respectively.

We focus on m.

<I> If m=1, the graph is K_n.

Since K_i, i=1, 2, 3, 4 is a planar graph. $\therefore$ t=1

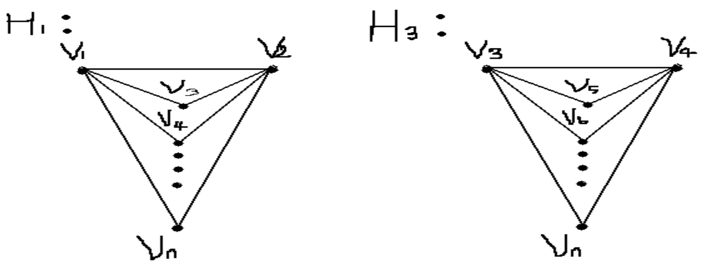
We consider n$\geq$5.

Since K_n is C_3-free, by previous result, the lower bound is $\left\lceil \frac{\binom{n}{2}}{3n-6} \right\rceil, n \geq 5$.

Let V(K_n)={$V_{1,}V_2,...,V_n$}, V(H_i)={$V_i,V_{i+1},...,V_n$},

E(H_i)={$V_iV_{i+1}, V_iV_j, V_{i+1}V_j \mid j = i+2, i+3, ..., n$}. $\forall i = 2k+1, k \in Z_{\left\lfloor \frac{n}{2} \right\rfloor}$.

Then K_n can decompose into H_i, i=2k+1, $k \in Z_{\left\lfloor \frac{n}{2} \right\rfloor}$. $\therefore$ The upper bound is $\left\lfloor \frac{n}{2} \right\rfloor$. For example,



Claim: $\left\lceil \frac{\binom{n}{2}}{3n-6} \right\rceil = \left\lfloor \frac{n+7}{6} \right\rfloor$

Since thickness must be an integer,

$$\Rightarrow \frac{n(n-1)}{n-2} = n(1+\frac{1}{n-2}) = n + \frac{n}{n-2} = n+1+\frac{2}{n-2}$$

$$\because \left\lceil \frac{x}{r} \right\rceil = \left\lfloor \frac{x+r-1}{r} \right\rfloor$$

$$\therefore \left\lceil \frac{n(n-1)}{6(n-2)} \right\rceil = \left\lceil \frac{1}{6}(n+1+\frac{2}{n-2}) \right\rceil = \left\lfloor \frac{n+1+\frac{2}{n-2}+6-1}{6} \right\rfloor = \left\lfloor \frac{n+7}{6} \right\rfloor$$

$$\therefore \left\lfloor \frac{n+7}{6} \right\rfloor \le t \le \left\lfloor \frac{n}{2} \right\rfloor, \textit{ for } n \ge 5 .$$

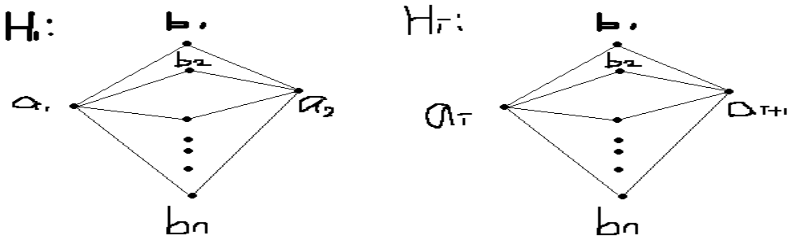
<II>If m=2, the graph is bipartite graph of each partite set of size n.

Since the graph is $C_3 - \textit{free}$, by previous result, the lower bound is

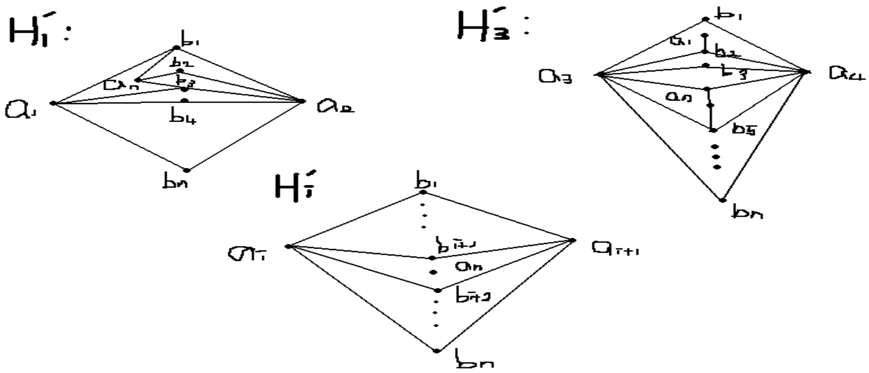
$$\left\lceil \frac{n^2}{2\cdot 2n-4} \right\rceil = \left\lceil \frac{n^2}{4n-4} \right\rceil .$$

Let one partite set is $\{ a_{1,\dots,}a_n \}$, another is $\{ b_{1,\dots,}b_n \}$.

n is even



n is odd



Let $V(H_i^{'}) = \{a_i, a_{i+1}, a_n, b_1, b_2, ..., b_n\}, i = 2k+1, k = 2, ..., \left\lfloor \frac{n}{2} \right\rfloor$

$$E(H_i^{'}) = \{a_i b_j, a_{i+1} b_j, a_n b_{i+1}, a_n b_{i+2} \mid j = 1, 2, ..., n\}$$

Then $K_{n,n}$ can be decomposed into $H_i^{'}, \forall i = 2k+1, k \in Z_{\left\lfloor \frac{n}{2} \right\rfloor}$

$\therefore$ Upper bound is $\left\lfloor \frac{n}{2} \right\rfloor$.But when n=4, and n=5, the thickness of

$K_{4,4}$ *and* $K_{5,5}$ is 2.

Hence if m=2 , we get the lower bound is $\left\lceil \frac{n^2}{4n-4} \right\rceil$, upper bound is $\left\lfloor \frac{n}{2} \right\rfloor$

when n$\geq$4.

<III>If m$\geq$3

If m is even.

We can see the multipartite graph as K_m with each partite set of size n. Then there exists (m-1)-1-factor and by the case m=2, we can know each 1-factor has an upper bound n.

Hence the multipartite graph has an upper bound (m-1)n.

If m is odd.

We delete one partite set, the remaining graph can be seen as K_{m-1}, Then there exist (m-2) 1-factor and each 1-factor has an upper bound n. then the (m-2) 1-factor has an upper bound (m-2)n.

When we put the deleting partite set back, we can see it as a complete bipartite graph $K_{1,m-1}$. So we need to draw additional n pages. Then the total upper bound is (m-2)n+n=(m-1)n.

Since the multipartite graph has a subgraph of C_3, by previous result, the lower bound is $\left\lceil \frac{\binom{m}{2} n^2}{3mn-6} \right\rceil$.

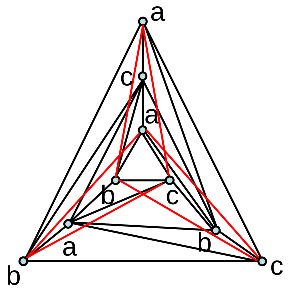
陳子鴻

4.Estimate the crossing number of $K_{3,3,3}$.

Sol:

Upper bound：

Since the following figure



, $\nu(K_{3,3,3}) \leqq 15$

Lower bound：

$$\nu(k_{3,6}) = \left\lfloor \frac{3}{2} \right\rfloor \left\lfloor \frac{3-1}{2} \right\rfloor \left\lfloor \frac{6-1}{2} \right\rfloor \left\lfloor \frac{6}{2} \right\rfloor = 6$$

Now, counting the size of the set as following

$\left\{(k_{3,6}, x) \mid x \text{ is a crossing point with the two edges in } k_{3,6},\ k_{3,6} < k_{3,3,3}\right\}$.

By two way counting:

1. $K_{3,6} \to x:\ \geq 3\times 6 = 18$; (A $K_{3,6}$ may belong to three different $K_{3,3,3}$.)

2. $x \to K_{3,6}:\ \leq \nu(K_{3,3,3})\times 2$ (A $K_{3,3,3}$ may contain two different $K_{3,6}$)

$\Rightarrow 18 \leq \nu(K_{3,3,3})\times 2 \Rightarrow \nu(K_{3,3,3}) \geq 9$.

Hence $9 \leq \nu(K_{3,3,3}) \leq 15$.

5. Find a 2-cell embedding of K_6 on the surface S_t for each $t \in \{2,3,4,5\}$.

(by Maple)

```
>#-------- input---------------
pi[1]:=[2,5,4,6,3,2];
pi[2]:=[1,3,4,5,6,1];
pi[3]:=[1,5,4,6,2,1];
pi[4]:=[1,2,3,5,6,1];
pi[5]:=[1,6,3,4,2,1];
pi[6]:=[1,2,3,5,4,1];
for i from 1 to 6 do
   for j from 1 to 5 do
      a[i][pi[i][j]]:=pi[i][j+1];
      a[i][i]:=0:
   end do;
end do;
> #--------indicator for arc(i,j)----------
for s from 1 to 6 do
   for t from 1 to 6 do
      b[s][t]:=0:
   end do;
  b[s][s]:=1:
end do:
#--------main program----------
r:=0:
for i from 1 to 6 do
   for j from 1 to 6 do
      s:=i;
      t:=j;
      if (b[s][t]=0) then
        print("start at edge",i,j);
      while (b[s][t]=0)
      do
        b[s][t]:=1:
        k:=a[t][s]:
        print(k);
        s:=t:
        t:=k:
      end do;
      r:=r+1;
      end if;
```

```
  end do;
end do;
print("r=",r);
```

$\gamma = 5$	$\gamma = 4$	$\gamma = 3$
$\pi_1 := [2, 3, 6, 5, 4, 2]$ $\pi_2 := [1, 3, 4, 5, 6, 1]$ $\pi_3 := [1, 2, 4, 5, 6, 1]$ $\pi_4 := [1, 2, 3, 5, 6, 1]$ $\pi_5 := [1, 2, 3, 4, 6, 1]$ $\pi_6 := [1, 2, 3, 4, 5, 1]$ "start at edge", 1, 2 3 4 5 6 1 5 2 6 3 1 6 2 1 3 2 4 3 5 4 6 5 1 4 2 5 3 6 4 1 2 "r=", 1	$\pi_1 := [2, 3, 4, 5, 6, 2]$ $\pi_2 := [1, 3, 4, 5, 6, 1]$ $\pi_3 := [1, 2, 4, 5, 6, 1]$ $\pi_4 := [1, 2, 3, 5, 6, 1]$ $\pi_5 := [1, 2, 3, 4, 6, 1]$ $\pi_6 := [1, 2, 3, 4, 5, 1]$ "start at edge", 1, 2 3 4 5 6 1 2 "start at edge", 1, 3 2 4 3 5 4 6 5 1 6 2 1 3 "start at edge", 1, 4 2 5 3 6 4 1 5 2 6 3 1 4 "r=", 3	$\pi_1 := [2, 5, 4, 6, 3, 2]$ $\pi_2 := [1, 3, 4, 5, 6, 1]$ $\pi_3 := [1, 5, 4, 6, 2, 1]$ $\pi_4 := [1, 2, 3, 5, 6, 1]$ $\pi_5 := [1, 6, 3, 4, 2, 1]$ $\pi_6 := [1, 2, 3, 5, 4, 1]$ "start at edge", 1, 2 3 1 2 "start at edge", 1, 3 5 4 6 1 3 "start at edge", 1, 4 2 5 1 4 "start at edge", 1, 5 6 4 1 6 2 1 5 "start at edge", 2, 4 3 6 5 3 4 5 2 6 3 2 4 "r=", 5

$\gamma = 2$

$\pi_1 := [2, 5, 4, 6, 3, 2]$

$\pi_2 := [1, 3, 4, 5, 6, 1]$

$\pi_3 := [1, 6, 5, 4, 2, 1]$

$\pi_4 := [1, 2, 3, 5, 6, 1]$

$\pi_5 := [1, 6, 4, 3, 2, 1]$

$\pi_6 := [1, 2, 3, 4, 5, 1]$

"start at edge", 1, 2

3 1 2

"start at edge", 1, 3

6 4 1 6 2 1 5 6 1 3

"start at edge", 1, 4

2 5 1 4

"start at edge", 2, 4

3 2 4

"start at edge", 2, 6

3 5 2 6

"start at edge", 3, 4

5 3 4

"start at edge", 4, 6

5 4 6

"r=", 7

$\gamma = 1$

$\pi_1 := [2, 6, 5, 4, 3, 2]$

$\pi_2 := [1, 3, 4, 5, 6, 1]$

$\pi_3 := [1, 6, 5, 4, 2, 1]$

$\pi_4 := [1, 2, 3, 5, 6, 1]$

$\pi_5 := [1, 6, 4, 3, 2, 1]$

$\pi_6 := [1, 2, 3, 4, 5, 1]$

"start at edge", 1, 2

3 1 2

"start at edge", 1, 3

6 4 1 3

"start at edge", 1, 4

2 5 1 4

"start at edge", 1, 5

6 1 5

"start at edge", 1, 6

2 1 6

"start at edge", 2, 4

3 2 4

"start at edge", 2, 6

3 5 2 6

"start at edge", 3, 4

5 3 4

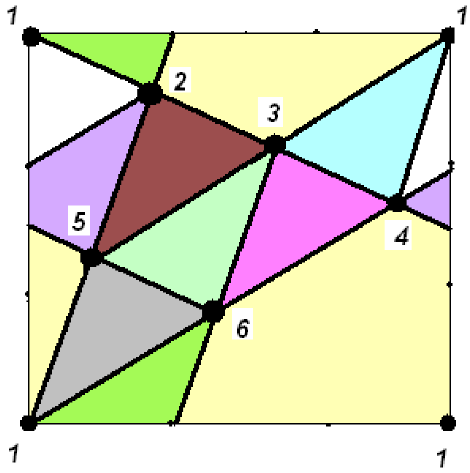
"start at edge", 4, 6

5 4 6

"r=", 9

以下是另一種 2-cell embedding

with 9 region on torus.



6. *Let the genus of G be n. Then, prove that any embedding of G on S_n is a 2-cell embedding.*

pf : Let the genus of G be n.

W.L.O.G. we may assume that all vertices of G lies on the sphere and G is connected. (∵ If G is disconnected, then the region between the components is not a 2-cell.)

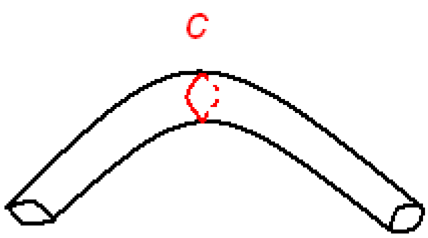
⇒ G won't have a 2-cell embedding.

Suppose there exists an embedding of G on S_n which is not a 2-cell embedding. So, there exists a region R, which is not a 2-cell. Let the boundary of R be C.

We have three cases to consider :

<Case1> *If R lies on one handle entirely.*

Since R is not a 2-cell, C is a cycle along the handle in this way:

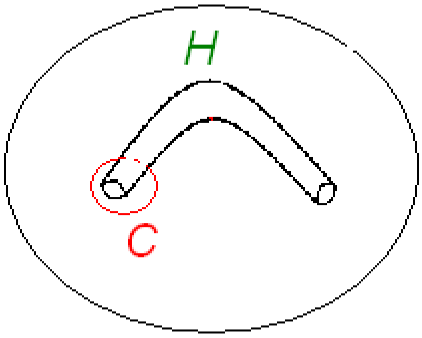


Hence, by cutting off the handle from C and paste two regions to the place we cut, we have an embedding of G on S_{n-1} (→←)

(Contradict to the genus of G is n)

<Case2> *If R lies on one sphere entirely.*

Since R is not a 2-cell, C is a cycle like this way:

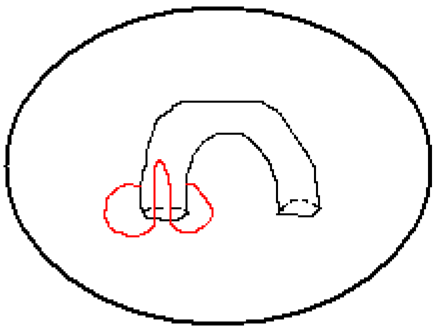


(C 在sphere 上,但不是2-cell,所以會圈住某個或多個handle 的一部分)

By making up the sphere to a handle and H to a sphere, then we return to case1. Hence we have an embedding of G on S_{n-1} (→←)

<u><Case3></u> *If R lies partially on sphere and partially on handle.*

Since R is not a 2-cell, C is a cycle like this way:

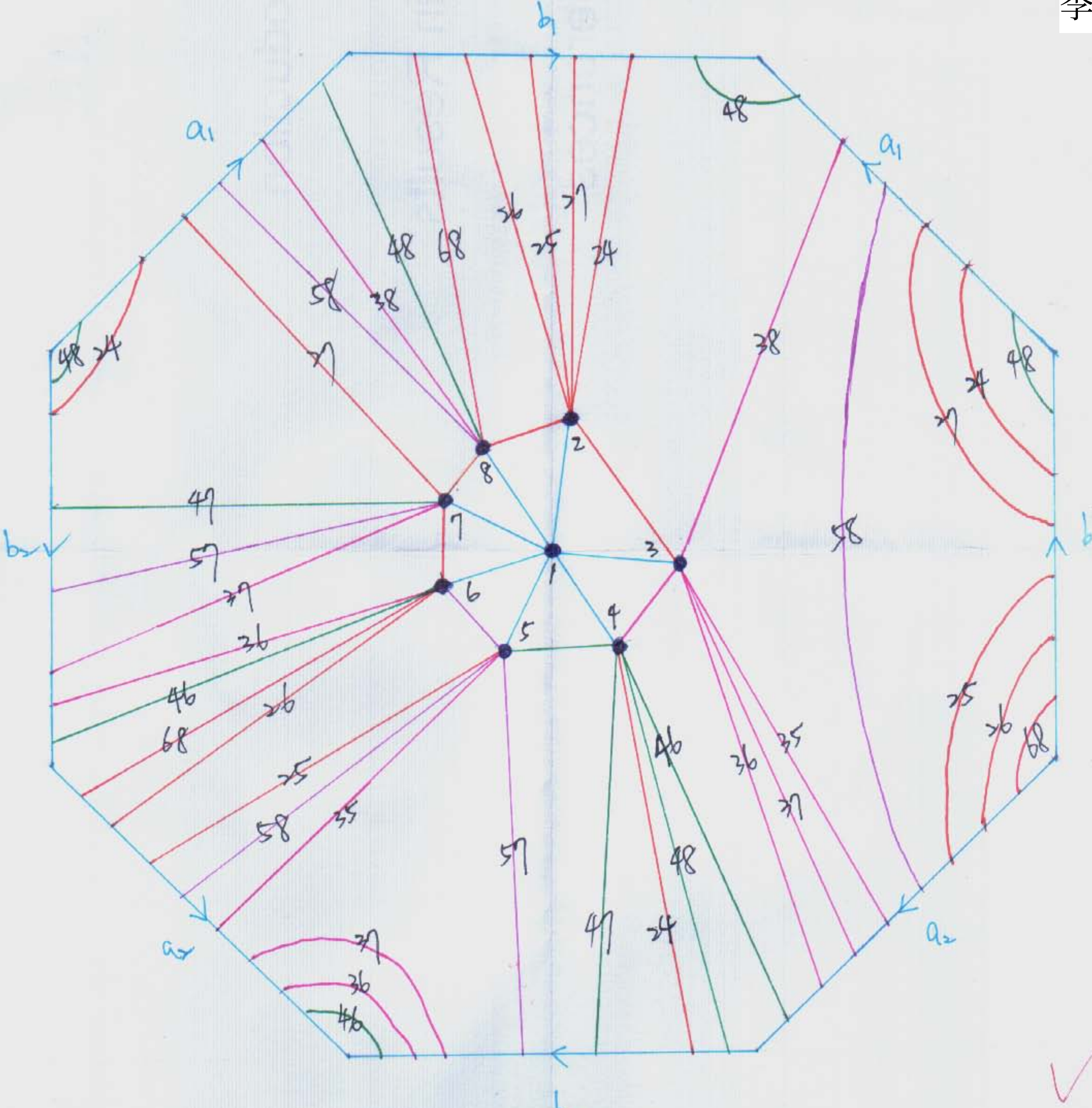


By redrawing the edges of C on the handle to the sphere, we have an embedding of G on S_{n-1} (→←)

By <case1> to <csse3> we have:

If the genus of G is n, then any embedding of G on S_n is a 2-cell embedding.

李光祥



棒!

Graph Theory II, Homework I 葉彬

Problem 8: Prove or disprove that K_7 has a 2-cell embedding on N_2.

Solution:

The generalized embedding scheme will be used here.

Let (P, λ) be a generalized embedding scheme of graph G, where P is a set of rotations, and λ is a function from the edge set of G, E(G), to the set {0,1}.

Consider the following simple example:

Let π_v = (e, d, f). This rotation is represented with edges rather than vertices for convenience. Suppose we start a region with different edges e_0, d_0, f_1, d_1, and then we have,

$\pi_v(e_0) = d_{(0+\lambda(d))}$ and $\pi_v(d_0) = f_{(0+\lambda(d))}$

$\pi_v(f_1) = d_{(1+\lambda(d))}$ and $\pi_v(d_1) = e_{(1+\lambda(d))}$

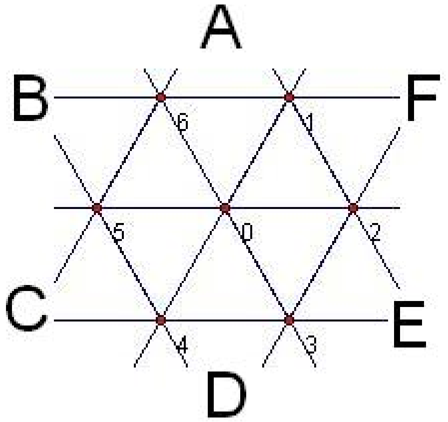
The addition is under modulo 2.

When $\lambda(e) = 0$ for all $e \in E(G)$, (P, λ) can be simplified to just P, the rotation scheme used on orientable surfaces. There are theorems [1] tells us every embedding scheme (P, λ) of a graph G determined a 2-cell embedding of G, and every 2-cell embedding of G is uniquely determined by some such scheme. Note that here the 2-cell embeddings include those 2-cell embeddings on non-orientable surfaces, i.e. every 2-cell embedding on a non-orientable surface can be represented when $\lambda(e) \neq 0$ for some $e \in E(G)$.

Thus, to prove K_7 has no 2-cell embedding on N_2, it is equal to prove that every 2-cell embedding of K_7 with 14 regions implies $\lambda(e) = 0$ for all $e \in E(K_7)$. The number of regions is from Euler-Poincaré formula, $p - q + r = 2 - h$, if the 2-cell embedding is on N_h. Therefore $7 - 21 + r = 2 - 2 \Rightarrow r = 14$.

Now $3r \leqq 2q$ since every region has at least 3 edges, but for K_7 on N_2 $3r = 42 = 2q$, thus we know every region for this embedding is triangle.

Let $V(K_7)$ = {0, 1, 2, 3, 4, 5, 6}. W.L.O.G. we may assume π_0 = (1, 2, 3, 4, 5, 6).



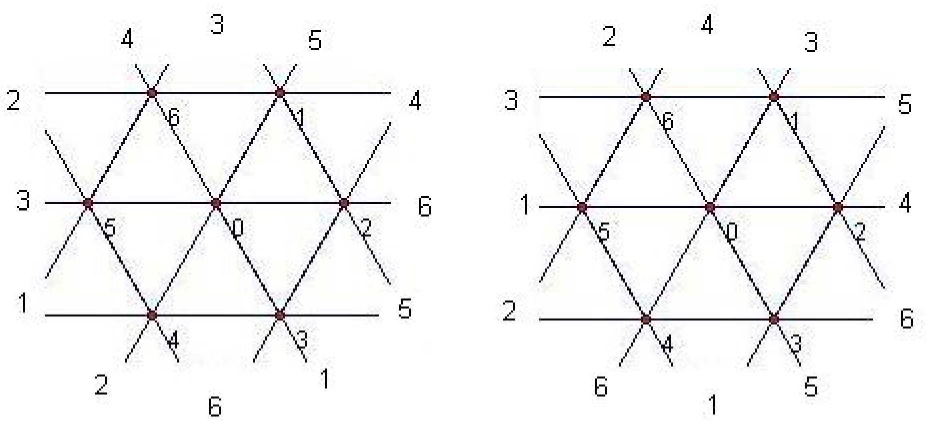
Thus the embeddings are:

π_0 = (1, 2, 3, 4, 5, 6)

π_1 = (2, 0, 6, …)

$\pi_2 = (3, 0, 1, \ldots)$
$\pi_3 = (4, 0, 2, \ldots)$
$\pi_4 = (5, 0, 3, \ldots)$
$\pi_5 = (6, 0, 4, \ldots)$
$\pi_6 = (1, 0, 5, \ldots)$

$\lambda(e) = 0$ for any edge e inside or on the boundary of this hexagon, since if not, a region which is not a triangle would be found.
Since all regions are triangle, A (and other upper-case letters) must be a vertex of {0, 1, 2, 3, 4, 5, 6}. A must be 3 or 4, because 1 and 6 already connect to 0, 2, 5. If we take A = 3, then B must be 2, and so on. The last neighbor of 6 is 4 since rest are determined. So we can complete the embedding in but only two ways as follow:



Now consider the left graph first. $\lambda(13) = 0$ since if not, then the successive edges 61 and 13 would be followed by the edge 35, which blocks the completion of a triangle. Similarly the edges 24, 35, 46, 51, and 62 must have the value 1. Then the $\lambda(36)$ must be 0 to complete the 61-13-36 triangle, so is 24 and 14, etc. Therefore we complete the whole proof for the left graph, and by same argument we can yield a similar proof for the right graph. Therefore K_7 has no embedding on N_2.

Graph Theory II, Homework I　　李柏瑩

9. Prove that $\chi(S_n)=\left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$ for each positive integer n.

Proof:

Step1

Since $\gamma(K_p)=\left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$

Set $p=\left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$

$\Rightarrow \gamma(K_p) \le n$

$\Rightarrow \chi(S_n) \ge \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$

Step2

Claim: If G is embeddable on S_n then $\exists v \in V(G)$ such that $\deg_G(v) \le c-1$ where c= $\frac{7+\sqrt{1+48n}}{2}$

proof: Case1: If $|G| \le c$

$\Rightarrow \delta(G) \le c-1$ we are done!

Case2: If $|G| = k > c$

Since we know that every simple k-vertex graph embedded on S_n has at most 3(k-2+2n) edges，the average degree satisfies

$$\frac{2\|G\|}{k} \leq \frac{6(k-2+2n)}{k}$$

$$= 6 - \frac{12-12n}{k}$$

$$\leq 6 - \frac{12-12n}{c} \text{ (Since k>c \& } n \in \text{)}$$

$$\underset{(*)}{=} \text{ c-1}$$

$\Rightarrow \exists v \in V(G)$ such that $\deg_G(v) \leq c-1$

Note:(*) holds

$\because$ c= $\frac{7+\sqrt{1+48n}}{2}$

$\Rightarrow 2c-7=\sqrt{1+48n}$

$\Rightarrow 4c^2-28c+49=1+48n$

$\Rightarrow 4c^2-28c+48-48n=0$

$\Rightarrow c^2-7c+12-12n=0$

$\Rightarrow c^2-c=6c-(12-12n)$

$\Rightarrow c-1=6-\frac{12-12n}{c}$

Step3

By induction on $|G|$

If $|G| \leq c$ then $\chi(G) \leq c$

Thus we only need to consider $|G| > c$

Assume it holds for $|G| = k-1$

Then for $|G| = k$ & G can be embedded on S_n

By Claim, we have a vertex $v \in V(G)$ such that $\deg_G(v) \leq c-1$

Now, let G' =G-v

$\Rightarrow$ G' =k-1

By induction hypothesis, $\chi(G') \leq c$

$\Rightarrow \chi(G) \leq c$ (Since $\deg_G(v) \leq c-1$)

Thus $\chi(S_n) \leq c = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$

Step1+Step2+Step3 $\Rightarrow \chi(S_n) = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$

Graph Theory II, Homework I　　連敏筠

10. $\gamma(Q_n) = (n-4)2^{n-3} + 1$

Proof:

$(\geq)$ $Since\ Q_n\ is\ a\ triangle - free\ (2^n, n2^{n-1})\ graph$

$$\begin{cases} p - q + r = 2 - 2r(G) \\ 4r \leq 2q \Rightarrow r \leq \frac{1}{2}q \end{cases}$$

$$p - q + \frac{1}{2}q \geq 2 - 2r(G)$$

$$r(G) \geq \frac{q}{4} - \frac{p}{2} + 1$$

$$\therefore r(Q_n) \geq \frac{n2^{n-1}}{4} - \frac{2^n}{2} + 1 = (n-4)2^{n-3} + 1.$$

$(\leq)$ By induction on n.

Define the statement A(n) as follows: The graph Q_n can be embedded on the surface of genus $(n-4)2^{n-3} + 1$ s.t the boundary of every region is 4-cycle and $\exists\, 2^{n-2}$ regions with pairwise disjoint boundaries.

A(2) and A(3) are true.

Assume A(k-1) is true, $k \geq 4$, let S be the surface of genus $(k-5)2^{k-4} + 1$ on which Q_{k1} is embedded.

Let Q_{k1} be embedded on another copy S' of the surface of genus $(k-5)2^{k-4} + 1$ s.t the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S. (that is, if v_1, v_2, v_3, v_4 are the vertices of the boundary of a region of Q_{k-1} on S, where the vertices list clockwise about the 4-cycle, then there is a region on S' with the vertices list counterclockwise.)

Now, consider 2^{k-3} distinguished regions of S together with the corresponding regions of S', and join each pair of associated regions by handle.

The addition of the first handle produces the surface of genus $2(k-5)2^{k-4}+1$ while the addition of each of the other $2^{k-3}-1$ handles results in an increase of one genus.

Now, each set of four vertices on the boundary of a distinguished region can be join to the corresponding four vertices on the boundary of the associated region s.t the four edges are embedded on the handle join the regions. The resulting graph is isomorphic to Q_k and every region is bounded by 4-cycle.

Each added handle gives rise to 4 regions "opposite" ones of which have disjoint boundaries, so exists 2^{k-2} regions of Q_k that are pairwise disjoint.

Thus A(n) is true for all n.