Chapter 8 Preliminaries of Algebraic Graph Theory

One of the main problems in Algebraic Graph Theory is to determine precisely how, or whether, properties of graphs are reflected in the algebraic properties of matrices associate with a graph such as adjacency matrix, incident matrix and the Laplacian.

§ 8.1. The Adjacency Matrix

Definition 8.1.1. The adjacency matrix $A(D)$ of a directed graph D is $A(D) = [d_{i,j}]_{n\times n}$ where $V(D) = \{v_1, v_2, ..., v_n\}$, $d_{i,j} = 1$ if and only if (v_i, v_j) is an arc of D. If D is a graph, then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0, 1)-$matrix.

Note that $d_{i,i} = 0$ for $i = 1, 2, ..., n$, since we consider graphs which have no multiple arcs(edges) and loops. The following result is a direct consequence of two isomorphic graphs(or digraphs).

Lemma 8.1.1. Let D_1 and D_2 are two digraphs defined on the same vertex-set of size n. Then $D_1 \cong D_2$ if and only if there is an $n \times n$ permutation matrix P such that $P^T A(D_1)P = A(D_2)$.

Corollary 8.1.2. If $D_1 \cong D_2$, then $A(D_1)$ and $A(D_2)$ are similar matrices.

Proof. This is a direct consequence of the fact that $P^T = P^{-1}$ for each permutation matrix. □

We can use the adjacency matrix of a digraph to find the number of different walks of length t from an assigned vertex to another assigned vertex.

Proposition 8.1.3 Let D be a digraph defined on $\{v_1, v_2, ..., v_n\}$. Let $A = [a_{i,j}] = A(D)$. Then, the number of different walks of length t from v_h to v_k is exactly the $(h, k)-$entry in A^t.
Proof. By induction on t. $\square$

Note. If G is a graph, then we also have a similar result. The following results can be obtained, but (4) is not easy at all.

Theorem 8.1.4. Let G be a graph with p vertices and q edges and $A = A(G)$. Then, we have
(1) the number of triangles in G is $\frac{1}{6}tr(A^3)$;
(2) the number of $4-$cycles in G is $\frac{1}{8}[tr(A^4) - 2q - 2\sum_{i\neq j} a_{i,j}^{(2)}]$ where $a_{i,j}^{(2)}$ is the $(i, j)-$entry in A^2;
(3) the number of $5-$cycles in G is $\frac{1}{10}[tr(A^5) - 5tr(A^3) - 5\sum_{i=1}^{p}\sum_{j=1}^{p}(a_{i,j} - 2)a_{i,i}^{(3)}]$; and
(4) the number of $6-$cycles in G is $\frac{1}{12}\{tr(A^6) - 6tr(A^4) + 5tr(A^3) - 4tr(A^2) - 3\sum_{i,j=1}^{p} a_{i,j}^{(3)} + 12\sum_{i,j=1}^{p} a_{i,j}^{(2)} - 3\sum_{i=1}^{p}[a_{i,i}^{(3)}]^2 + 9\sum_{i\neq j} a_{i,j}^{(2)}(a_{i,j}^{(2)} - 1)a_{i,j}^{(1)} - 6\sum_{i\neq j} a_{i,j}^{(2)}(a_{i,j}^{(2)} - 1)(a_{i,i}^{(2)} - 2) - 2\sum_{i=1}^{p} a_{i,i}^{(2)}(a_{i,i}^{(2)} - 1)(a_{i,i}^{(2)} - 2)\}$.
Proof. Omitted.

Now, we consider the eigenvalues of $A(D)$.

Definition 8.1.2. Let A be an $n \times n$ matrix. The characteristic polynomial of A is the polynomial $\phi(A, x) = det(xI - A)$. For convenience, we use $\phi(D, x)$ to denote the The characteristic polynomial of $A(D)$.

Definition 8.1.3. The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. For convenience, we also use "the spectrum of a graph G" to denote the spectrum of $A(G)$. (The eigenvalues of A are the zreos of $det(xI - A)$.)
Note. The eigenvalues of A are the numbers λ satisfying $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x} \in \mathbb{R}^n$ if A is an $n \times n$ matrix. The eigenspace of an eigenvalue λ of A is the set $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}$, denoted by $\varepsilon(\lambda)$.

e.g. Let G be a $4-$cycle. Then $\phi(G, x) = x^4 - 4x^2$. This implies that the eigenvalues are: $2, 0, 0, -2$; and $(1, 1, 1, 1)^T$, $(1, 1, -1, -1)^T$, $(-1, 1, 1, -1)^T$, $(1, -1, 1, -1)^T$ are their corresponding eigenvectors respectively.

Proposition 8.1.5. All eigenvalues of a graph G (simple) are real.
Proof. This follows from the fact that $A(G)$ is a symmetric matrix. □

Definition 8.1.4. The largest eigenvalue of a graph G is called the index of G.

Therefore, the index of C_4 is 2.

Theorem 8.1.6. A non-negative matrix always has a non-negative eigenvalue r such that the modulus of any other eigenvalue does not exceed r. To this "maximal" eigenvalue there corresponds a non-negative eigenvector (all components are non-negative).
Proof. (Refer to Matrix Theory.)

Corollary 8.1.7. For each graph G, there exist a non-negative eigenvector corresponds to the index of G.

For convenience, we shall use $Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_t \\ m_1 & m_2 & \dots & m_t \end{pmatrix}$ to denote the spectrum of G where m_i is the multiplicity λ_i. In what follows we shall list some properties related to $Spec(G)$.

• $\sum_{i=1}^{t} m_i\lambda_i = tr(A)$, $A = A(G)$. If we list all eigenvalues as $\lambda_1, \lambda_2, ..., \lambda_n$, then $\sum_{i=1}^{n} \lambda_i = tr(A)$. It follows from $det(xI - A) = \prod_{i=1}^{n}(x - \lambda_i)$.

• $\prod_{i=1}^{n} \lambda_i = det(A)$. This obtained by letting $x = 0$.

• The multiplicity of an eigenvalue λ is $n - rank(\lambda I - A)$. This is obtained by finding the dimension of $\varepsilon(\lambda)$.

To determine $Spec(G)$ of a graph G is in general very difficult. For special graphs, we are able to find all the eigencalues.

• $Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

Note that if G is an $r-$regular graph, then r is an eigenvalue of G with one of its eigenvectors $(1, 1, ..., 1)^T$.

• $Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$.

The following result of bipartite graphs is worth of mention here.

Theorem 8.1.8. The following statements are equivalent.
(1) G is bipartite.
(2) The nonzero eigenvalues of G occur in pairs λ_i, λ_j with $\lambda_i = -\lambda_j$.
(3) $\phi(G, x)$ is a polynomial in x^2.
Proof. This is mainly by the fact that if λ is an eigenvalue of G with multiplicity m, so is $-\lambda$. □

Now, we are ready to study the relationship between eigenvalues and graph parameters.

Theorem 8.1.9. The diameter of a connected graph G is less than the number of distinct eigenvalues.
Proof. Note that the number of r distinct eigenvalues is equal to the degree of minimal polynomial of A. Therefore, some linear combination of $A^0 = I, A^1, A^2, ..., A^n$ is 0. This implies that if $A^0, A^1, ..., A^k$ are

linearly independent, then $k < n$.
Now, assume that $diam(G) = k$. Hence, there exist two vertices v_i, v_j such that $d(v_i, v_j) = k$ and thus $A^{(k)}_{i,j} \neq 0$. But, $A^{(t)}_{i,j} = 0$ for each $t < k$ by the choose of v_i and v_j. So, $A^O, A^1, ..., A^k$ are linear independent.(?) $\square$

The following two lemmas are very useful in obtaining upper bounds.

Lemma 8.1.10. If G' is an induced subgraph of G, then $\lambda_{min}(G) \leq \lambda_{min}(G') \leq \lambda_{max}(G') \leq \lambda_{max}(G)$ where $\lambda_{min}(G)$ and $\lambda_{max}(G)$ are the minimum and maximum eigenvalues of G respectively.
Proof. By Linear Algebra, the maximum and minimum eigenvalues of a real symmetric matrix can be attained by using $\vec{x}A\vec{x}$ defined on the set of unit vectors $\vec{x}$, i.e., $\lambda_{min}(A) \leq \vec{x}A\vec{x} \leq \lambda_{max}(A)$. Since G' is an induced subgraph of G, $A' = A(G')$ can be assumed to be an upper left principal submatrix of $A = A(G)$ (by permuting the vertex set). So, let $\vec{z'}$ be the unit eigenvector of $\lambda_{max}(G')$ and thus $A\vec{z'} = \lambda_{max}(G')\vec{z'}$. Now, by letting $\vec{z'} =< \vec{z'}, \vec{0} >^T$, we have $\vec{z}^T A\vec{z} = \vec{z'}^T A'\vec{z'} = \lambda_{max}(G')$. Since $\vec{z}^T A\vec{z} \leq \lambda_{max}(G)$, we have the last inequality. Similarly, we have the first inequality and the proof follows. $\square$

Lemma 8.1.11. For every graph G, $\delta(G) \leq \frac{2\|G\|}{|G|} \leq \lambda_{max}(G) \leq \Delta(G)$.
Proof. We prove the last inequality. Let $\vec{x}$ be an eigenvector for eigenvalue λ, and let $x_j = max\{x_i | \vec{x} = (x_1, x_2, ..., x_n)\}$. Then $\lambda x_j = (A\vec{x})_j = \sum_{v_i \in N(v_j)} x_i \leq d(v_j)x_j \leq \Delta(G)x_j$. This implies that $\lambda_{max}(G) \leq \Delta(G)$. $\square$

Now, we can improve Brook's Theorem a little bit.

Theorem 8.1.12. (Wilf, 1976)
For every graph G, $\chi(G) \leq 1 + \lambda_{max}(G)$.

Proof. Let $\chi(G) = k$ and H be a (vertex-critical) subgraph of G such that $\forall v \in V(H)$, $\chi(H) = k$ and $\chi(H-v) = k-1$. Then $\delta(H) \geq k-1$.(?) Therefore, $k \leq 1+\delta(H) \leq 1+\lambda_{max}(H) \leq \lambda_{max}(G)$. □

Let J be the $n \times n$ all $1's$ matrix. Then, we have a necessary and sufficient condition to characterize a connected regular graph.

Theorem 8.1.13. A graph G is regular and connected if and only if J is a linear combination of powers of $A(G)$.
Proof. ($\Leftarrow$) This is a direct consequence of the assumption which can be checked easily.
($\Rightarrow$) Since G is $k-$regular, k is an eigenvalue of $A(G)$. Therefore, the minimal polynomial of $A(G)$ is $f(G;\lambda) = (\lambda - k)g(\lambda)$. Since $f(G; A(G)) = 0$, $Ag(A) = kg(A)$. This implies that each column of $g(A)$ is an eigenvector of $A(G)$ for the eigenvalue k. By the fact that G is connected and regular, each column of $g(A)$ is a multiple of $(1, 1, ..., 1)^T$. Since $g(A)$ is a symmetric matrix, all columns are identical vectors and thus a multiple of J. This concludes the proof. □

In fact, we can say more about connected regular graphs.

Theorem 8.1.14. If a simple graph G is $k-$regular, then G and $\overline{G}$ have the same eigenvectors. Moreover, the eigenvalue associated with $\vec{1}$ is k in G and $n-k-1$ in $\overline{G}$. In general, if $\vec{x}$ is a nonconstant eigenvector of G for λ of G, then its associated eigenvalue in $\overline{G}$ is $-1-\lambda$.
Proof. By direct checking. □

Definition 8.1.5. (Strongly Regular Graph)
A strongly regular graph $srg(v, k, \lambda, \mu)$ is a graph G which has the following properties:
(1) $|G| = v$, (2) G is $k-$regular, (3) If $x \sim_G y$, then x and y have exactly λ common neighbors, and (4) If $x \nsim_G y$, then x and y have

exactly μ common neighbors.

e.g. A $5-$cycle is an $srg(5, 2, 0, 1)$ and the Petersen graph is an $srg(10, 3, 0, 1)$.

Theorem 8.1.15. Let A be the adjacency matrix of G. Then G is an $srg(v, k, \lambda, \mu)$ if and only if (a) $AJ = kJ$ and (b) $A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$ where J is the all $1's$ matrix of order $|G|$.
Proof. ($\Rightarrow$) (a) follows by the fact that G is $k-$regular. (b) follows by the definition of a strongly regular graph(?). (For example, consider the (i, i) entry of the left-hand side matrix, $l_{i,i}$. $l_{i,i} = k + (\mu - \lambda) \cdot 0 + (\mu - k) = \mu$. Clearly, the corresponding entry of the right-hand side matrix is also μ.)
($\Leftarrow$) This is also easy to see by verifying (1) $\sim$ (4) of an $srg(v, k, \lambda, \mu)$. □

Note that finding strongly regular graphs with parameters v, k, λ and μ is not an easy task, it is one the main topics of Algebraic Graph Theory.

§ 8.2. The Incidence Matrix

Let G be a graph definded on V with edge set E. Furthermore, let $|V| = |G| = p$ and $|E| = \|G\| = q$, and $V = \{v_1, v_2, ..., v_p\}$ and $E = \{e_1, e_2, ..., e_q\}$. Then, we can define the incidence matrix of G.

Definition 8.2.1. (Incidence Matrix)
An incidence matrix of $G = (V, E)$ is a $p \times q$ matrix $B(G) = [b_{i,j}]$ such that each entry of $B(G)$ is either 0 or 1 and $b_{i,j} = 1$ if and only if $v_i \in e_j$.

Note that the definition is also well-defined when G is a hypergraph. Since a design $(\mathbb{X}, \mathbb{B})$ can be recognized as a hypergraph, the incidence matrix of a design can be defined accordingly.

In what follows, we use B to denote $B(G)$ and B^T to denote the transpose of B. The following result is not difficult to see.

Theorem 8.2.1. Let G be a $(p, q)-$graph. Then, $A(L(G)) = B^T B - 2I_q$.

Proof. Notice that in $B^T B = [l_{i,j}]_{q\times q}$, $l_{i,i}$ is the inner product of the i^{th} column of B with itself and therefore $l_{i,i} = 2$ since each edge contains two vertices. If $i \neq j$, then $l_{i,j}$ is the inner product of the i^{th} column and the j^{th} column, therefore $l_{i,j} = 1$ if $e_i \cap e_j \neq \emptyset$ and $l_{i,j} = 0$ otherwise. Now, the proof follows by checking the $(i, j)-$entry of $A(L(G))$. □

Incidence matrices have many applications. By considering each edge as a set of a vertices, we can define a very important notion in group testing.

Definition 8.2.2. (Disjunct Matrices)
Let the set of rows of a matrix M be indexed by $\{1, 2, ..., n\}$ and the set of columns be indexed by $\{S_1, S_2, ..., S_m\}$. Then, the matrix $M_{n\times m}$ is said to be d-disjunct if for each $i \in \{1, 2, ..., m\}$, S_i is not contained

in the union of at most d other columns.

Clearly, the incidence matrix of a simple graph is 1-disjunct. For more informations, the readers may refer to the boos "Combinatorial Graph Testing" by Du and Hwang. (堵丁柱與黃光明教授)

§ 8.3. Laplacian of a graph

In order to explore the interaction between Spectral Graph Theory and Differential Geometry, we need this notion.

Definition 8.3.1. (Laplacian)
The Laplacian $L = [l_{i,j}]$ of a graph G is defined as follows:

$$l_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } deg_G(v_i) \neq 0; \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \sim_G v_j \text{ and } d_i = deg_G(v_i); \text{and} \\ 0 & \text{otherwise .} \end{cases}$$

Now, let $L = [a_{i,j}]$ where (a) $a_{i,j} = d_i$ if $i = j$, (b) $a_{i,j} = -1$ if $v_i \sim_G v_j$ and (c) $a_{i,j} = 0$ otherwise. Then, we have

Proposition 8.3.1. $L = T^{-\frac{1}{2}} L T^{-\frac{1}{2}}$ where T is the diagonal matrix with the (i,i)-entry having value d_i.
Proof. A routine matter to check. □

By using the Laplacian of a graph G, we are able to obtain the following results.

Theorem 8.3.2. Let G be a graph of order n and its eigenvalues are $\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then,
(1) $\sum_{i=1}^{n} \lambda_i \leq n$ with equality holding if and only if G has no isolated vertices.
(2) For $n \geq 2$, $\lambda_1 \leq \frac{n}{n-1}$ with equality holding if and only if G is the complete graph of order n.
(3) If G has no isolated vertices, then $\lambda_{n-1} \geq \frac{n}{n-1}$.
(4) If G is not a complete graph, then $\lambda_1 \leq 1$.
(5) If G is connected, then $\lambda_1 > 0$. Moreover, if $\lambda_i = 0$ and $\lambda_{i+1} \neq 0$, then G has $i + 1$ components.
(6) For $i \leq n - 1$, $\lambda_i \leq 2$.

(7) $\lambda_{n-1} = 2$ if and only if a (connected) component of G is bipartite.

(8) The spectrum of a graph is the union of the spectra of its components.

Proof. Not trivial at all. We leave the proofs to the readers. $\square$