Chapter 6 Extremal Graphs

§ 6.1. **Basic Notions**

Definition 6.1.1(Forbidden graphs)

A graph F is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, G does not contain F as a subgraph.
e.g. K_5 and $K_{3,3}$ are forbidden graphs of planar graphs.

Definition 6.1.2

$ex(n; F) = \max\{\|G\| \mid G$ is a graph of order n such that F is not a subgraph of $G\}$

Definition 6.1.3(Extremal graphs with forbidden graph F)

The graph G of order n with $\|G\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F.
e.g. Let $F = K_3$. Then $ex(n; F) = \lfloor \frac{n^2}{4} \rfloor$ and $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the unique extremal graph of order n forbids K_3.

We may forbid a class of graphs $\mathcal{F}$ instead of just one.

Theorem 6.1.1. Let $\mathcal{F}$ be the class of odd cycles. Then $ex(n;\mathcal{F})=\lfloor \frac{n^2}{4} \rfloor$.

Theorem 6.1.2.(*Pósa*)

Let G be a graph with $|G| \geq 3$ and $deg_G(x) + deg_g(y) \geq k$ for each pair of non-adjacent vertices x and y in G. Then, for $k < n$ G has a path of length k and a cycle of length at least $\frac{k+2}{2}$, and for $k = n$, G has a Hamiltonian cycle.

Proof. Review Chapter 2. □

Theorem 6.1.3. Let $k \leq n$ and $n = tk + s, 0 \leq s < k$. Then $ex(n; P_{k+1}) \leq \frac{(k+1)n}{2}$ and $K_k \bigcup K_k \bigcup ... \bigcup K_k \bigcup K_k$(t-tuples) is an extremal graph of order n forbids P_{k+1}.

Proof. By induction on n.

First, if G is disconnected, then let A be a component of G and $B = G - A$. Then, by induction $ex(|A|; P_{k+1}) = \frac{(k-1)|A|}{2}$ and $ex(|B|; P_{k+1}) = \frac{(k-1)|B|}{2}$. This implies that $ex(n; P_{k+1}) = \frac{(k-1)n}{2}$. On the other hand, if G is connected, then G can not contain a proper subgraph K_k and there exists a vertex x such that $deg_G(x) \leq \frac{k-1}{2}$. For, otherwise, G has a path of length k.
Now, consider $G - x$.$G - x$ can not contain a path of length k, hence $\|G\| = deg_G(x) + \|G - x\| \leq \frac{k-1}{2} + \frac{(k-1)(n-1)}{2} = \frac{(k-1)n}{2}$.
The fact that $K_k \bigcup ... \bigcup K_k \bigcup K_k$ is an extremal graph is easy to see.

□

Definition 6.1.4($Turán$ Graph)
For $n \geq r \geq 1$, $T_r(n) =_{def} K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n}{r} \rfloor, ..., \lceil \frac{n}{r} \rceil}$.

Lemma 6.1.4. Let G be a complete r-partite graph of order n.
Then $\|G\| \leq \|T_r(n)\|$.
Proof Easy to see.

Corollary 6.1.5. Let G be an r-partite graph of order n.
Then $\|G\| \leq \|T_r(n)\|$.

Theorem 6.1.6.($Turán$ Theorem)
$ex(n; K_{r+1}) = \|T_r(n)\|$ and $T_r(n)$ is the unique extremal graph.

Proof. Since $T_r(n)$ does not contain a subgraph K_{r+1},
$ex(n; K_{r+1}) \geq \|T_r(n)\|$. We claim $ex(n; K_{r+1}) \leq \|T_r(n)\|$. By Lemma 6.1.4, it suffices to show that if G does not contain a subgraph K_{r+1}, then there exists an r-partite graph H such that $\|G\| \leq \|H\|$.
By induction on r and it is clear that $r = 1$ (and $r = 2$) is true. Let G be a graph with maximum degree $\Delta(G)$ and G does not contain a subgraph K_{r+1}. Let $x \in V(G)$ s.t. $deg_G(x) = \Delta(G)$. Now, consider $G' = < N_G(x) >_G$. Clearly, this graph does not contain K_r as a subgraph. Therefore, there exists an $(r-1)$-partite graph H' such that $\|H'\| \leq \|G'\|$. Let $S = V(G) \setminus V(H')$. Then, the graph $H' \vee S$ (join) is an r-partite graph which has $\|H'\| + (n - \Delta(G))\Delta(G)$ edges. Clearly, it is larger than $\|G'\| + \Sigma_{v \in S} deg_G(v) \geq \|G\|$. This concludes the proof that $\|T_r(n)\| \geq \|G\|$.

On the other part, let $y \in V(G)$, $deg_G(y) = \delta(G)$. The proof is also by induction. Then $\|G - y\| = \|G\| - deg_G(y) \geq \|T_r(n-1)\|$,

$G-y$ must be isomorphic to $T_r(n-1)$. This implies that in $G-y$ the smallest partite set has $\lfloor \frac{n-1}{r} \rfloor$ vertices. Hence, $|< y, G-y >| = n-1-\lfloor \frac{n-1}{r} \rfloor = n-\lceil \frac{n}{r} \rceil$. Now, y must be non-adjacent to at least one partite set, or, we have a K_{r+1}, since $G-y$ is $T_r(n-1)$. Moreover, the partite set must be of size $\lfloor \frac{n-1}{r} \rfloor$, hence $G \cong T_r(n)$. □

Proof. 2 (by *Turań*):

Since G does not contain K_{r+1} as a subgraph, we may add edges to G to contain $\tilde{G}$ s.t. $K_r \leq \tilde{G}$. Let $V(K_r) = W$ and $V(G)\backslash W = U$. Now,

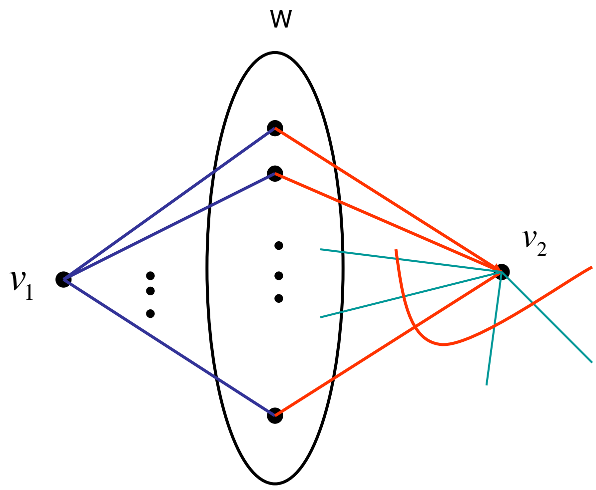
$$\begin{aligned} \|E(G)\| &\leq \binom{r}{2} + (r-1)(n-r) + \| < U >_G \| \\ &\leq \binom{r}{2} + (r-1)(n-r) + \|T_r(n-r)\| \\ &= \|T_r(n)\|. \end{aligned}$$

□

Proof. 3 (by Zykov):

Let $v_1 \in V(G)$ s.t. $deg_G(v_1) = \Delta(G)$ and $W = N(v_1)$. ($\|W\| = \Delta(G)$). Let $\tilde{G} = G - < N(v_1) >_G + T_{r-1}(\Delta(G))$. Consider $U_1 = V(G_1)\backslash(W \cup \{v_1\})$. If this is an empty set, then stop. Otherwise, let $v_2 \in U_1$. Delete all edges in G_1 which are incident to v_2 and add v_2x to G_1 for each $x \in W$. Let this new graph be G_2. Now $|E(G_2)| = \|G_2\| \geq \|G_1\| \geq \|G\|$ and G_2 does not contain K_{r+1} as a subgraph. The process is continued if we have vertices other than vertices in W and v_2. Finally, we obtain a complete r-partite graph H and conclude that $\|G\| \leq \|H\| \leq \|T_r(n)\|$. □

(Note) The process of deleting the edges incident to v_2 and let v_2 join to all the vertices in W is called a symmetrization of v_2 to v_1, see the following figure.



Stronger version of *Turán*'s Theorem

Theorem 6.1.7.(*Erdös*, 1970)

Let G be a graph of order n which does not contain K_{r+1} as a subgraph. Then, there exists an H satisfying
(1) H is an r-partite graph,
(2) $V(H) = V(G)$, and
(3) $\forall x \in V(G)$, $deg_G(x) \leq deg_H(x)$.

Moreover, if G is not a complete r-partite graph, then there exists a vertex $z \in V(G)$, s.t. $deg_G(z) < deg_H(z)$.

Proof. By induction on "r" and $r = 1$ is clearly true.
Let the assertion be true for $r' < r$. Let $x \in V(G)$ s.t. $deg_G(x) = \Delta(G)$. Let $W = N_G(x)$. Then $< W >_G = G_0$ does not contain K_r. Therefore, there exists an $(r-1)$-partite graph H_0, s.t. $V(H_0) = W$ and $\forall y \in W$, $deg_{G_0}(y) \leq deg_{H_0}(y)$, moreover, if G_0 is not a complete $(r-1)$-partite graph, then $\exists y' \in W$, s.t. $deg_{G_0}(y') < deg_{H_0}(y')$. Now, let $V = V(G)$ and $H = H_0 \vee (V \backslash W)$. Clearly, H is an r-partite graph. Now, $\forall z \in V(G)$,
(1) $z \in V \backslash W$, $deg_G(z) \leq \Delta(G) = deg_H(z)$ $(= \|W\|)$,
(2) $z \in W$, $deg_G(z) \leq deg_{G_0}(z) + n - |W| \leq deg_{H_0}(z) + n - |W| = deg_H(z)$.

To show the final part of the statement, we assume that for each $y \in V(G)$, $d_G(y) = d_H(y)$, hence $\|G\| = \|H\|$. This implies that $\|G_0\| = \|H_0\|$.

In fact, $d_{G_0}(x) = d_{H_0}$ $\forall x \in W$. For otherwise, let $deg_{G_0}(x') <$

$d_{H_0}(x')$. Then $d_G(x') < d_H(x') = d_{H_0} + n - |W|$, we are done.($d_G(x') \leq d_{G_0}(x') + n - |W|$). This implies that G_0 is a complete $(r-1)$-partite graph, by induction. $\square$

In what follows, we study $ex(n; K_{s,t})$. First, we consider a famous problem.

Zarankiewicz Problem

Let $m \geq s$ and $n \geq t$. Determine the maximum number of edges in a bipartite $G \leq K_{m,n}$ such that $K_{s,t} \not\leq G$. Denote this value by $z(m, n; s, t)$.

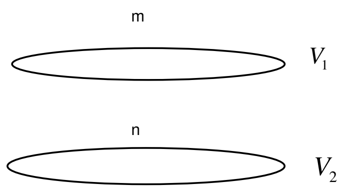
(*) Extremal graphs which are restricted to bipartite graphs.
e.g. $z(7, 7; 2, 2) \geq 21$. (?)

Definition 6.1.5.(Convex function f)

A function $f : \mathbb{R} \to \mathbb{R}$ is convex if
$tf(x) + (1-t)f(y) \geq f(xt + y(1-t))$, $0 \leq t \leq 1$.

Lemma 6.1.8. Let $2 \leq s \leq m$, $2 \leq t \leq n$, $0 \leq r \leq m$, $z = km+r$,and $z = my$. Let $G_2(m, n)$ be a graph of size z which does not a subgraph $K_{s,t}$. Then $m\binom{y}{t} \leq (m-r)\binom{k}{t} + r\binom{k+1}{t} \leq (s-1)\binom{n}{t}$.

Proof. Let $G_2(m, n) = (V_1, V_2)$.



Define $H := (V_1, \binom{V_2}{t})$ where $\binom{V_2}{t} =_{def}$ {t-subsets of V_2}.
Let $A \in \binom{V_2}{t}$. Then $\forall x \in V_1$, $x \sim_H A$ iff $x \sim_G a, \forall a \in A$.

$\Rightarrow \|H\| = \Sigma_{x \in V_1}\binom{deg_G(x)}{t}$.

(*) $\forall B \in \binom{V_2}{t}, deg_H(B) \leq s-1$.

Otherwise, $\exists$ s vertices in V_1, which are joining to every vertex of B. Then, we have a $K_{s,t}$.) The implies that $\|H\| \leq (s-1)\binom{n}{t}$, i.e. $\|H\| = \Sigma_{x \in V_1}\binom{deg_G(x)}{t} \leq (s-1)\binom{n}{t}$. Now, since $\Sigma_{x \in V_1} deg_g(x) = z = km + r = my$, we have

$m\binom{y}{t} \leq (m-r)\binom{k}{t} + r\binom{k+1}{t} \leq \Sigma_{x \in V_1}\binom{deg_G(x)}{t} \leq (s-1)\binom{n}{t}$. $\square$

Theorem 6.1.9. $z(m,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$.

Proof. Let $G = G_2(m,n)$ forbid $K_{s,t}$, and $\|G\| = z(m,n;s,t) = my$. Since $m\binom{y}{t} \le (s-1)\binom{n}{t}$, $\frac{\binom{y}{t}}{\binom{n}{t}} \le \frac{s-1}{m}$. Hence,
$\frac{y(y-1)...(y-t+1)}{n(n-1)...(n-t+1)} \le \frac{s-1}{m}$, $\forall 0 \le i \le t-1$.

$$\begin{aligned}\frac{y-i}{n-i} - \frac{y-t+1}{n-t+1} &= \frac{(y-i)(n-t+1)-(n-i)(y-t+1)}{(n-i)(n-t+1)} \\ &= i(y-n) + (t-1)(n-y) \\ &= (n-y)(t-1-i) \ge 0\end{aligned}$$

This implies that $\frac{(y-t+1)^t}{(n-t+1)^t} \le \frac{s-1}{m}$.

$$\begin{aligned}&\Rightarrow (y-t+1)^t \le (s-1)(n-t+1)^t m^{-1}. \\ &\Rightarrow y-t+1 \le (s-1)^{\frac{1}{t}}(n-t+1)m^{\frac{-1}{t}}. \\ &\Rightarrow my \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m.\end{aligned}$$

□

Corollary 6.1.10. $z(n,n;s,z) \le \frac{n}{2}[1+(4(s-1)(n-1)+1)^{\frac{1}{2}}]$.

Proof. Since $n\binom{y}{2} \le (s-1)\binom{n}{2}$.

$$ny(y-1) \le (s-1)n(n-1),$$

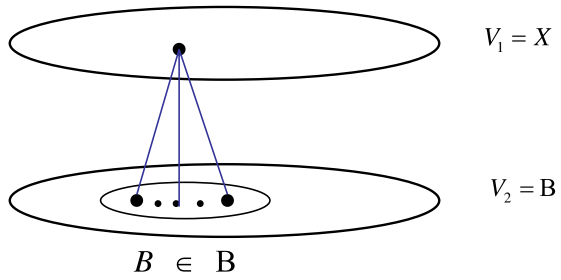
$$y^2 - y - (s-1)(n-1) \le 0, \qquad y \le \frac{1+[1+4(s-1)(n-1)]^{\frac{1}{2}}}{2}.$$

□

Corollary 6.1.11. $z(n,n;2,2) \le \frac{n}{2}[1+(4n-3)^{\frac{1}{2}}]$, equality holds when $n = q^2+q+1$, q is a prime power.

Proof. Since q is a prime power, there exists a projective plane of order q. Let it be (X,B). Review that $|X| = |B|$. Let $G_{X,B}$ be as following :

$(G_{X,B})$



Contain no $K_{2,2}$. $(|B| = q+1)$.

$\|G_{X,B}\| = (q+1)(q^2+q+1)$.
Since $n = q^2+q+1$, $q = \frac{1}{2}[-1+(4n-3)^{\frac{1}{2}}]$. This implies that
$z(n,n;2,2) \geq (q^2+q+1)(q+1) = \frac{n}{2}[1+(4n-3)^{\frac{1}{2}}]$. □

Theorem 6.1.12. $ex(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$.

Proof. $z(n,n;s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)n^{1-\frac{1}{t}} + (t-1)n$

$$= (s-1)^{\frac{1}{t}}(n^{2-\frac{1}{t}} - n^{1-\frac{1}{t}}(t-1)) + (t-1)n$$

$$\leq (s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + (t-1)n.$$

Next, we claim $ex(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + O(n)$.
Let H be an extremal graph and $V(H) = \{x_1, x_2, ..., x_n\}$. Now, define a bipartite graph $G = (V_1, V_2)$ such that $V_1 = \{x'_1, x'_2, ..., x'_n\}$, and $V_2 = \{x''_1, x''_2, ..., x''_n\}$ and $x'_i \sim_G x''_j$ if and only if $x_i \sim_H x_j$. Since $G \not\geq K_{s,t}$, $H \not\geq K_{s,t}$. By the fact $z(n,n;s,t) \leq (s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + (t-1)n$,
$\|H\| \leq (s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + (t-1)n$.
Since $\|G\| = \frac{1}{2}\|H\|$, $\|G\| \leq \frac{1}{2}(s-1)^{\frac{1}{2}} n^{2-\frac{1}{t}} + (t-1)n$.
This concludes the proof. □

Another proof of $(t \leq s)$ $ex(n; K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + O(n)$.

Proof. We count $K_{1,t}$'s in G. Since $G \not\geq K_{s,t}$, every set of t-vertices is the t-set of at most $s-1$ t stars with distinct centers. So, the total number of possible number of t stars is $(s-1)\binom{n}{t}$. On the other hand, let the degree sequence of G be $< d_1, d_2, ..., d_n >$, then the number of distinct t stars, $K_{1,t}$, is equal to $\Sigma_{i=1}^{n}\binom{d_i}{t}$. Now, let $\|G\| = m$, then

$\Sigma_{i=1}^{n} d_i = 2m$. This implies that $n\binom{\frac{2m}{n}}{t} \leq \Sigma_{i=1}^{n}\binom{d_i}{t} \leq (s-1)\binom{n}{t}$.

$$\begin{aligned}
&\Rightarrow n\frac{2m}{n}(\frac{2m}{n}-1)...(\frac{2m}{n}-t+1) \leq (s-1)n(n-1)...(n-t+1)\\
&\Rightarrow \quad (\frac{\frac{2m}{n}-t+1}{n-t+1})^{t} \leq \frac{s-1}{n}\\
&\Rightarrow \quad \frac{2m}{n}-t+1 \leq (n-t+1)(s-1)^{\frac{1}{t}}n^{\frac{-1}{t}}\\
&\Rightarrow m \leq \frac{1}{2}(n-t+1)(s-1)^{\frac{1}{t}}n^{1-\frac{1}{t}} + \frac{1}{2}(t-1)n\\
&\qquad \leq \frac{1}{2}(s-1)^{\frac{1}{t}}n^{2-\frac{1}{t}} + O(n).
\end{aligned}$$

□

e.g. $ex(n;C_4) = ex(n;K_{2,2}) \leq \frac{n}{4}(1+\sqrt{4n-3})$ and equality holds whenever $n = q^2+q+1$, q is a prime power.

(*) Problem : Find extremal graphs in $G_{m,n}$ (not containing $K_{2,2}$).

Definition 6.1.6 A graph G is said to be H**-saturated** if $G \ngeq H$ and $\forall e \in E(\bar{G})$, $G+e \geq H$.

(Fact) An extremal graph not containing H as a subgraph is H-**saturated**. But, not necessary "the other implication".

A star of size larger than two is C_3-*saturated.*

A 5 - cycle is also C_3-*saturated.*

Exercise. Find a minimal graph of order n which is C_4-*saturated.*

Problem. Find a graph of order n with minimum number of edges which is C_4-*saturated.*

In what follows, we shall apply the idea of design theory to obtain a beautiful solution of the above problem.

Definition 6.1.7. $(\mathbf{X}, \mathbb{B})$ is a pairwise balanced design (PBD) with $\lambda = 1$ if for each pair of distinct elements (2-subsets) of X occurs in exactly one block in $\mathbb{B}$. $(\mathbf{X}, \mathbb{B})$ is also denoted by $2-(v, K, 1)$ design where $|\mathbf{X}| = v$, $K = \{|B| \mid B \in \mathbb{B}\}$.
e.g. $\mathbb{B} = \{X, \{x\}, \{x\}, ..., \{x\}\}$ is a $2-(|\mathbf{X}|, \{|\mathbf{X}|, 1\}, 1)$ design.

Definition 6.1.8. We define $G_{\mathbf{X},\mathbb{B}}$ as a bipartite graph $(\mathbf{X}, \mathbb{B})$ where $x \sim_{G_{\mathbf{X},\mathbb{B}}} B$ iff $x \in B$. Therefore, $\|G_{\mathbf{X},\mathbb{B}}\| = \Sigma_{B\in\mathbb{B}}|B|$.

Lemma 6.1.13. $G_{\mathbf{X},\mathbb{B}}$ is a C_4-*saturated* graph provided that $(\mathbf{X}, \mathbb{B})$ is a $2-(\mathbf{X}, K, 1)$ design.

Proof. Let $e = (x, B)$ be an arbitrary edge not in $G_{\mathbf{X},\mathbb{B}}$. Hence, $x \notin B$. Since $B \neq \phi$, let $y \in B$. Then the pair $\{x, y\}$ occurs twice after adding e to $G_{\mathbf{X},\mathbb{B}}$. ($\{x, y\} \subseteq N_{\mathbf{X},\mathbb{B}}(B')$ for some B'.) □

A **partial** $2-(|\mathbf{X}|, K, 1)$ design is an $(\mathbf{X}, \mathbb{B}')$ such that every pair of distinct elements of $\mathbf{X}$ occurs in **at most one** block in $\mathbb{B}'$.

Definition 6.1.9. A partial $2-(v, K, 1)$ design $(\mathbf{X}, \mathbb{B}'')$ is said to be an **extension** of a partial $2-(v, K, 1)$ design $(\mathbf{X}, \mathbb{B}')$ if $\mathbb{B}'' \neq \mathbb{B}'$, $|\mathbb{B}'| = |\mathbb{B}''|$ and $B'_i \subseteq B''_i$ for $i = 1, 2$ where $B'_i \in \mathbb{B}'$ and $B''_i \in \mathbb{B}''$.

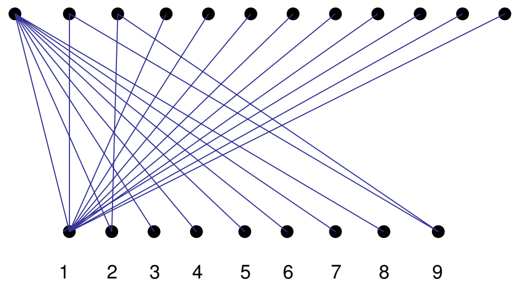
Definition 6.1.10. A partial 2-$(v, K, 1)$ design $(\mathbf{X}, \mathbb{B}')$ is said to be non-extendable if it has no extension.
e.g. $\mathbf{X} = \{1, 2, ..., 9\}$

$$\mathbb{B}_1 = \{\{1, 2, ..., 9\}, \underbrace{\{1\}, \{1\}, ..., \{1\}}_{11 \text{ sets}}\} = \{B_1, B_2, ..., B_{12}\}.$$

$(\mathbf{X}, \mathbb{B}_1)$ is non-extendable. $\Sigma_{i=1}^{12}|B_i| = 20$.
e.g. $\mathbf{X} = \{1, 2, ..., 9\}$,

$$\mathbb{B}_2 = \{\{1, 2, ..., 8\}, \{1, 9\}, \{2, 9\}, \underbrace{\{1\}, ..., \{1\}}_{9 \text{ sets}}\} = \{B_1, B_2, ..., B_{12}\}.$$

$(\mathbf{X}, \mathbb{B}_2)$ is non-extendable, $\Sigma_{i=1}^{12}|B_i| = 21$.



Maximize $\Sigma_{i=1}^{|\mathbb{B}|}|B_i|$ such that $(\mathbf{X}, \mathbb{B})$ is a non-extendable 2-(v,K,1) design.

$\mathbb{B} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\{2,5,8\},\{3,6,9\},\{1,5,8\},$ $\{2,6,7\},\{3,4,8\},\{1,6,8\},\{2,4,9\},\{3,5,7\}\}$.
$(\mathbf{X}, \mathbb{B})$ is in fact an affine plane of order 3.
$\Sigma_{B\in\mathbb{B}}|B| = 36$.

Can we find a larger one ? No!!

Theorem 6.1.14. Let $(\mathbf{X}, \mathbb{B})$ be a 2-$(m, K, 1)$ design s.t. $|\mathbf{X}| = m$, $|\mathbb{B}| = n$, $K = \{k, k+1\}$ and $\binom{m}{2} - \Sigma_{B\in\mathbb{B}}\binom{|B|}{2} < k$. Then $G_{\mathbf{X},\mathbb{B}}$ is an extremal graph in $G_{m,n}$ which does not contain C_4. ($G_{\mathbf{X},\mathbb{B}}$ is C_4-*saturated*).

Proof. By definition $\|G_{\mathbf{X},\mathbb{B}}\| = \Sigma_{B\in\mathbb{B}}|B|$. Let $\mathbb{B} = \{B_1, B_2, ..., B_n\}$ and $\mathbf{X} = \{v_1, v_2, ..., v_m\}$. Moreover, let $|B_i| = x_i$, i=1,2,...,n.
Suppose that there exists another PBD $(\mathbf{X}, \mathbb{A})$ such that $\mathbb{A} = \{A_1, A_2, ..., A_n\}$

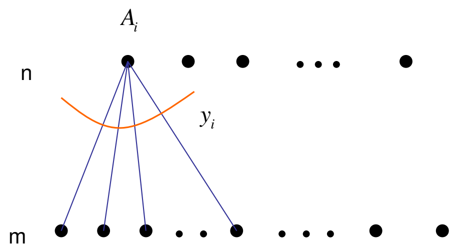
$$\Sigma_{A\in\mathbb{A}}|A| > \Sigma_{B\in\mathbb{B}}|B| \text{ i.e.,} \Sigma_{i=1}^{n}|A_i| > \Sigma_{i=1}^{n}|B_i|.$$

Let $|A_i| = y_i$. Then $\Sigma_{i=1}^{n}x_i < \Sigma_{i=1}^{n}$.
Let $y_i - x_i = d_i$ and $\binom{y_i}{2} - \binom{x_i}{2} = d_i^*$, i=1,2,...,n.
Consider

$$\begin{aligned} d_i^* &= \binom{y_i}{2} - \binom{x_i}{2} = \frac{y_i(y_i-1)}{2} - \frac{x_i(x_i-1)}{2} = \frac{y_i^2 - x_i^2 + x_i - y_i}{2} \\ &= \frac{(y_i - x_i)(y_i + x_i - 1)}{2} = (y_i - x_i)x_i + \frac{(y_i - x_i)(y_i - x_i - 1)}{2} \\ &\geq d_i x_i \geq d_i k. \end{aligned}$$

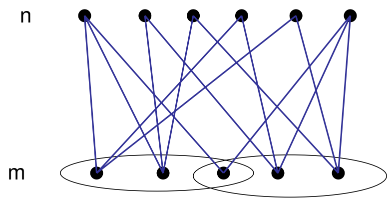
(By induction)
This implies that $\binom{m}{2} - \Sigma_{B\in\mathbb{B}}\binom{|B|}{2}$

$$= \binom{m}{2} - \Sigma_{i=1}^{n}\binom{x_i}{2} = \binom{m}{2} - \Sigma_{i=1}^{n}(\binom{y_i}{2} - d_i^*)$$
$$= \binom{m}{2} - \Sigma_{i=1}^{n}\binom{y_i}{2} + \Sigma_{i=1}^{n}d_i^*$$
$$\geq \binom{m}{2} - \Sigma_{i=1}^{n}\binom{y_i}{2} + \Sigma_{i=1}^{n}d_i k.(\binom{m}{2} - \Sigma_{i=1}^{n}\binom{y_i}{2} \geq 0)$$
$$\geq k(\Sigma_{i=1}^{n}d_i)$$
$$\geq k(\Sigma_{i=1}^{n}d_i \geq 1). \longrightarrow\longleftarrow .$$



□

e.g. $k = 2$,



$\binom{m}{2} = \binom{5}{2} = 10, \Sigma_{i=1}^{n}\binom{|B_i|}{2} = 3+1+1+1+1+3 = 10.$

So far, we have obtained, $ex(n; K_{r+1})$ and an estimation of $ex(n; K_{s,t})$. Therefore, it is natural to ask the question that when H is a balanced complete multipartite graph what is the value $ex(n; H)$. Review that if $H \cong K_{r+1}$, then

$$lim_{n\to\infty} ex(n; K_{r+1})/\binom{n}{2} = (1 - \frac{1}{r}). \quad (1)$$

(1) can be easily checked by considering $n = qr$.

$$\|T_n(n)\| = \binom{n}{2} - r\binom{q}{2} = \binom{n}{2} - \frac{rq(q-1)}{2} = \binom{n}{2} - \frac{rq(rq-r)}{2r}$$
$$\geq \binom{n}{2} - \frac{1}{r}\binom{rq}{2} = (1-\frac{1}{r})\binom{n}{2}.$$

Lemma 6.1.15. $\forall \varepsilon > 0$, for sufficiently large n, every graph G of order n with $\delta(G) \geq \varepsilon n$, G contain a subgraph $K_{t,t}$ where $t \geq \varepsilon logn$.

Proof. Let $t = \lceil \varepsilon logn \rceil$. Since $\delta(G) \geq \varepsilon n$, for each vertex $v \in V(G)$, v is adjacent to at least $\binom{\varepsilon n}{t}$ distinct t-subsets. It suffices to prove that $n\binom{\varepsilon n}{t} \geq t\binom{n}{t}$.(?) This can be obtained by the following inequalities.

$$\frac{t\binom{n}{t}}{n\binom{\varepsilon n}{t}} = \frac{t \cdot n(n-1)...(n-t+1)}{n \cdot \varepsilon n(\varepsilon n-1)...(\varepsilon n-t+1)}$$
$$= \frac{t}{n}\varepsilon^{-t} \cdot \frac{n(n-1)...(n-t+1)}{n(n-\frac{1}{\varepsilon})...(n-\frac{t-1}{\varepsilon})}$$
$$\leq \frac{t}{n}\varepsilon^{-t}(\frac{n}{n-\frac{t-1}{\varepsilon}})^t = \frac{t}{n}\varepsilon^{-t}(1-\frac{t-1}{n\varepsilon})^{-t}$$
$$\leq ... \leq \frac{2t}{\varepsilon n}n^{(log\frac{1}{\varepsilon})^{\varepsilon}} < 1 (n \to \infty).$$

□

Now, we are ready to prove the most well-known theorem in Extremal Graph Theory.

Theorem 6.1.16.(*Erdös* and *Stone*, 1946)
$\forall \varepsilon > 0$, $\exists n_0 = n_0(r, \varepsilon)$, s.t. if $|G| = n \geq n_0$ and $\delta(G) \geq (1+\varepsilon-\frac{1}{r})n$, then $K_{r+1(t)} \geq G$ where $t \geq \frac{\varepsilon logn}{2^{r-1}(r-1)!}$.

Proof. By induction on r.
If $r = 1$, then the assertion is true by above lemma. Assume that the assertion is true for "$r < r'$" (including $r' = r-1$), $r \geq 2$, $|G| = n$ and $\delta(G) \geq (1+\varepsilon-\frac{1}{r}n)$. (let $\varepsilon = \frac{1}{r(r+1)}$) Then $\delta(G) \geq (1+\varepsilon-\frac{1}{r})n > (1-\frac{1}{r})n = (1-\frac{1}{r-1}+\frac{1}{r(r-1)})n$ where $r' = r-1$ and $\varepsilon = \frac{1}{r(r-1)}$, hence $G \geq K_{r(t')}$ where $t' \geq \frac{\varepsilon logn}{2^{r-2}(r-2)!}$. For convenience, let $d_r = \frac{1}{2^{r-2} \cdot r!}$, then t' may be chosen as $\lceil d_r logn \rceil$.

Now, let $K_{r(t')} = K \geq G$ and $U \subseteq V(G)\backslash V(K)$ such that $\forall x \in U$, $|< x, K >| \geq (1 - \frac{1}{r} + \frac{\varepsilon}{2})|K|$. (Now, we count how many vertices dose H have !)
First, consider $|< K, G - K >| = f$,

$|K|[(1-\frac{1}{r}+\varepsilon)n-|K|] \leq f \leq |U||K|+(|G|-|U|-|K|)(1-\frac{1}{r}+\frac{\varepsilon}{2})|K|$.
(vertices which connect to K in $G-U-K$ is less than $(1-\frac{1}{r}+\frac{\varepsilon}{2})|K|$)
$\Rightarrow \frac{r\varepsilon n}{2} \leq |U|(1-\frac{r\varepsilon}{2}) + |K|(\frac{r\varepsilon}{2} - 1)$.
As $n \to \infty$, ε arbitrarily small, $\frac{r\varepsilon n}{2} \leq |U|$, and thus $|U| \geq \varepsilon n$. Now, let $t = \lceil \frac{\varepsilon logn}{2^{r-1}(r-1)!} \rceil$, then $t \geq \lceil \frac{r\varepsilon}{2} t' \rceil$. Since $|K| = rt'$,
$\lceil (1 - \frac{1}{r} + \frac{\varepsilon}{2})|K| \rceil = \lceil (r-1)t' + \frac{r\varepsilon}{2} t' \rceil \geq (r-1)t' + t$.
This implies that $\forall x \in U$, x has at least $(r-1)t' + t$ neighbors in K.
By the fact that $t' \geq t$, $< N(x) >_K \supseteq K_{r(t)} = G'$. Thus, $G \geq K_{r+1(t)}$.
(?) □

(**Another Version** of *Erdös* and Stone's Theorem)

Theorem 6.1.17. For all integers $r \geq 2$ and $t \geq 1$, and every $\varepsilon > 0$, there exists an integer n_0 such that every graph G of order $n \geq n_0$ and $\|G\| \geq \|T_r(n)\| = \varepsilon n^2$, $G \geq K_{(r+1)(t)}$.

(*)**Theorem 6.1.18**. For every graph H with at least one degree,
$lim_{n\to\infty} ex(n;H) \cdot \binom{n}{2}^{-1} = \frac{\mathcal{X}(H)-2}{\mathcal{X}-1}$.

Proof. Review that $lim_{n\to\infty} \|T_r(n)\| \binom{n}{2}^{-1} = \frac{r-1}{r}$.
Let $\mathcal{X}(H) = r+1$. Then $\forall n$, $H \nleq T_r(n)$, since H is not r-colorable. Therefore, $ex(n;H) \geq \|T_r(n)\|$. On the other hand, $H \leq K_{(r+1)(t)}$ for some t (large enough). $ex(n;H) \leq \|T_r(n)\| + \varepsilon n^2$. (By E.S. Thm). Hence,

$$\|T_r(n)\| \leq ex(n;H)\binom{n}{2}^{-1}$$

$$\leq (\|T_r(n)\| + \varepsilon n^2)\binom{n}{2}^{-1}$$

$$= \|T_r(n)\|\binom{n}{2}^{-1} + \varepsilon n^2/\binom{n}{2}$$

$$= \|T_r(n)\|\binom{n}{2}^{-1} + 2\varepsilon/(1-\frac{1}{n})$$

$$\leq \|T_r(n)\|\binom{n}{2}^{-1} + 4\varepsilon(n \geq 2).$$

Since $lim_{n\to\infty}\|T_r(n)\|\binom{n}{2}^{-1} = 1 - \frac{1}{r}$, $lim_{n\to\infty}ex(n;H)\binom{n}{2}^{-1} = 1 - \frac{1}{r}$. This implies that $lim_{n\to\infty}ex(n;H)\binom{n}{2}^{-1} = \frac{r-1}{r} = \frac{\mathcal{X}(H)-2}{\mathcal{X}(H)-1}$. □

§ 6.2. Ramsey Theory

Definition 6.2.1. The Ramsey number $R(s,t)$ is the smallest value of n for which either a graph G of order n contains a subgraph K_s or $K_t \leq \bar{G}$.

Definition 6.2.2. $R(s,t)$ is the smallest value of n for which every red-blue edge-coloring of K_n yields a red K_s or a blue K_t.

Definition 6.2.3. We use $R(s)$ to denote $R(s,s)$ and in this situation, every red-blue edge-coloring of K_n yields a monochromatic K_s.

Theorem 6.2.1. $R(s,2) = R(2,s) = s$, $R(s,t) = R(t,s)$. For $s > 2$ and $t > 2$, $R(s,t) \leq R(s,t-1)+R(s-1,t)$..(1)
$R(s,t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$..(2)

Proof. Claim of (1).

Let $n = R(s,t-1) + R(s-1,t)$ and K_n be red-blue colored. Since for each vertex $v \in K_n$, $deg(v) = n-1$. Let x be a vertex of K_n. Consider the edges incident to x. By Pigeon-Hole principle, either there are $R(s,t-1)$ blue edges or $R(s-1,t)$ red edges. If there are $n_1 = R(s,t-1)$ blue edges, then the red-blue colored complete graph K_{n_1} induced by the neighbors of x through blue edges contains either a red K_s or a blue K_{t-1}. In either case, K_n contains a red K_s or a blue K_t. Similarly, the other case.

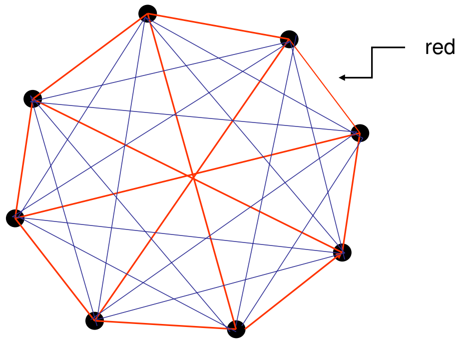
Claim of (2).

Clearly, it is true whenever one of s and t is equal to 2. Assume that $s > 2$, $t > 2$ and (2) holds for all pairs (s', t') with $2 \leq s' + t' < s + t$. Then, by $R(s,t) \leq R(s-1,t) + R(s,t-1)$,
$R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$. □

Corollary 6.2.2. $R(s,t)$ is finite for all finite pairs (s,t).

Corollary 6.2.3. $R(s,t) \leq R(s,t-1)+R(s-1,t)-1$ if both $R(s,t-1)$ and $R(s-1,t)$ are even.

Example $R(3,4) = 9$.



$\Rightarrow R(3,4) \geq 9$.
$R(3,4) \leq 9(?)$.

$R(i,j)$

$R(i,j)$:

	2	3	4	5	…					
2	2	3	4	5	6	7	8	9		
3	3	6	9	14	18	23	28	36		
4	4	9	18	25						
⋮	5	14	⋮	…						
	6									

Since,

$$R(s,s) = R(s) \leq \binom{2s-2}{s-1} = \frac{(2s-2)(2s-3)\ldots(2s-2-s+2)}{(s-1)!}$$
$$= \frac{(2s-2)(2s-3)\ldots(s)(s-1)!}{(s-1)!(s-1)!},$$

we have $\frac{(2s-2)!}{(s-1)!^2} \leq \frac{2^{2s-2}}{\sqrt{s}}$.

This implies
Theorem 6.2.4(*Thomason*, 1988).
For each $s \geq 2$, $R(s) \leq \frac{2^{2s}}{s}$.

Theorem 6.2.5. $R(p_1, p_2, ..., p_t) \leq R(p_1 - 1, p_2, ..., p_t) + R(p_1, p_2 - 1, p_3, ..., p_t) + ... + R(p_1, p_2, ..., p_t - 1) - t + 2$.

Corollary 6.2.6. $R(3,3,3) \leq R(2,3,3) + R(3,2,3) + R(3,3,2) - 1 = 17$.

Theorem 6.2.7. $R(3,3,3) = 17$.

Proof. From Corollary 6.2.6, it suffices to find a graph coloring (edge-colored) of K_{16} with three colors such that no monochromatic triangle exists.

$V(K_{16}) = \mathbb{Z}_{16}$

$i \sim j$ is colored c_t if and only if $i - j \equiv t \ (mod 3)$ where t is taking 1,2,3. □

- $R_k(3)$ is finite for each fixed finite positive integer.

Proposition 6.2.8. $R_k(3) \leq \lfloor e(k-1)! \rfloor + 1$.

Proof. By induction on k and it's true for $k = 2$ and 3. Let $k > 3$. By hypothesis, $R_{k-1}(3) \leq \lfloor e(k-1)! \rfloor + 1$.
Moreover, $R_k(3) \leq kR_{k-1}(3) - k + 2$.

$$\begin{aligned} &= k(\lfloor e(k-1)! \rfloor + 1) - k + 2 = k\lfloor e(k-1)! \rfloor + 2 \\ &= \lfloor e(k)! \rfloor - 1 + 2 \leq \lfloor ek! \rfloor + 1. \end{aligned}$$

□

(Note) $\lfloor ek! \rfloor = 1 + k\lfloor e(k-1)! \rfloor$.(check !)

Theorem 6.2.9. (Schur, 1916)
For every $k \geq 1$ there is an integer m such that every k-coloring of $[1, m]$ contains integers x, y and z of the same color such that $x+y = z$.

Proof. Let $m = R_k(3) - 1 = R(\underbrace{3, 3, ..., 3}(k - tuples))$ and $n = R_k(3)$. Now, given a coloring c of $[1, m]$, i.e., $c : [1, m] \to [1, k]$, we obtain an induced coloring c' of K_n by letting $V(K_n) = \mathbb{Z}_n$ and $c'(ij) = c(|i-j|)$. Since $n = R_k(3)$, there exists a monochromatic C_3, $\{h, i, j\}, n \geq j > i > h \geq 1$ in K_n. Now, let $x = i - h$, $y = j - i$ and $z = j - h$, we have $x + y = z$. □

Theorem 6.2.10. (van der Waerden, 1927)

Given p and k, if n is large enough, then every k-coloring of $[1, n]$ contains a monochromatic arithmetic progression of length p.

van der Waerden functions $W(p)$ and $W(p, k)$

$W(p) = W(p, 2)$

$W(p, k)$ is the minimal value of n that if $[1, n] = \cup_{i=1}^{k} N_i$, then there are a and $d \geq 1$, and $1 \leq i(some) \leq k$ s.t.
$a, a + d, ..., a + (p - 1)d \in N_i$.

$W(2) = 3$
$W(3) = 9$
$W(4) = 35$
$W(5) = 178....?$
$W(p, 3) =?$

Proof of van der Waerden's Theorem (Exercise)

Definition 6.2.4. A sequence of natural numbers $s_1, s_2, ..., s_l$ is said to be super-increasing if for each $1 \leq i < l$, $s_1 + s_2 + ... + s_i < s_{i+1}$.
(Fact) $0 \leq \Sigma_{i=1}^{l} \varepsilon_i s_i \leq \Sigma_{i=1}^{l} s_i$ where $\varepsilon_i \in \{0, 1\}$ for $i = 1, 2, ..., l$.

$f((\varepsilon_1, \varepsilon_2, ..., \varepsilon_l)) = f(\varepsilon_1, \varepsilon_2, ..., \varepsilon_l) = \Sigma_{i=1}^{l} \varepsilon_i s_i$ defined a 1-1 function from $\{0, 1\}^l$ into $\mathbb{N}$.

Definition 6.2.5. Let $s_0 \in \mathbb{N}$. Then $C = C(s_0; s_1, s_2, ..., s_l) = \{s_0 + \Sigma_{i=1}^{l} \varepsilon_i s_i \mid \varepsilon_i \in \{0, 1\}\}$ is called an $l - cube$ in $\mathbb{N}$ provided that
$< s_1, s_2, ..., s_l >$ is a super-increasing sequence of natural numbers.

Theorem 6.2.11. (Hilbert, 1892)

If $\mathbb{N}$ is colored with finitely many colors, then, for every $l \geq 1$, one of the color classes contains infinitely many translates of the same $l - cube$.

Proof. We claim that there is a function $H : \mathbb{N}^2 \to \mathbb{N}$ such that if $N_0 \geq H(k, l)$ then every k-coloring of $[1, N_0]$ contains a monochromatic $l - cube$.

[Note that since N_0 is a finite natural number, the proof follows by using the k-coloring to N_0.]

The proof is by induction on l. Since a 1-cube in $\mathbb{N}$ is simply a pair of

two distinct natural numbers, we let $H(k,1) = k+1$. Then, the proof follows by using Pigeon-Hole principle.
(Claim) If $H(k,l) \leq n$, then $H(k,l+1) = N_0 = kn^{l+1}$.

Now, let φ be a k-coloring of $[1, N_0] = [1, kn^{l+1}]$, i.e., $\varphi : [1, N_0] \to [1, k]$. Partition $[1, N_0]$ into $N_0/n = kn^l$ intervals $I_1, I_2, ..., I_{kn^l}$, where $I_j = [(j-1)n+1, jn]$, $j = 1, 2, ..., N_0/n$. Since $H(k,l) \leq n$, each interval I_j contains a monochromatic $l-cube$ by induction hypothesis. Now, each interval has n elements and thus each interval contains at most $(n-1)^l$ distinct $l-cubes$ (up to **translations**). Therefore, there are at least $\lceil \frac{kn^l}{(n-1)^l} \rceil \geq k+1$ $l-cubes$ which are the same (up to translation). By P.H.P., at least two of them are of the same color. Now, the proof follows by combining these two $l-cubes$ together, which is a $(l+1)-cube$. □

Theorem 6.2.12 (Ramsey)

Let A be an infinite set. Then, for each $k(\geq 1)$ coloring of the elements in $\binom{A}{r}$, there exists a monochromatic $\binom{T}{r}$ where T is also an infinite set.

Proof. The proof is by induction on r and it is clear that the assertion is true when $r = 1$. Now, assume that the assertion is true for the value smaller than r $(r \geq 2)$.

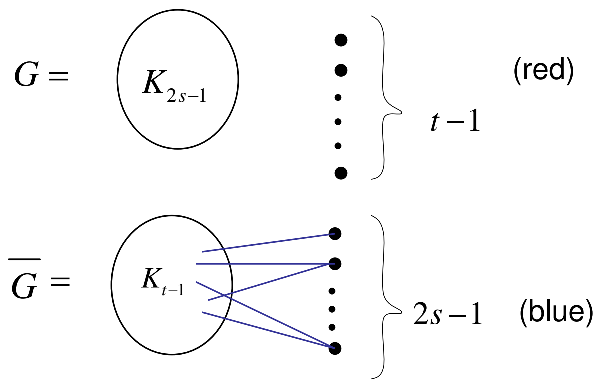
Let φ be a k-coloring of $\binom{A}{r}$ and $A = A_0$. Let $x_1 \in A_0$ and $B_1 = A_0 \backslash \{x_1\}$. Define a coloring φ' of $\binom{B_1}{r-1}$ as follows : $\forall S \in \binom{B_1}{r-1}$, $\varphi'(S) = \varphi(S \cup \{x_1\})$. Clearly, φ' is a k-coloring of $\binom{B_1}{r-1}$ and thus there exists a subset of B_1, A_1, such that $\binom{A_1}{r-1}$ is monochromatic. Following the same process, we have that $\binom{A_i}{r-1}$ is monochromatic for $i \in \mathbb{N}$. Furthermore, $A_1 \supseteq A_2 \supseteq ....$ Since we only use k colors, there exists a color which occurs in $\binom{A_{i_1}}{r-1}$, $\binom{A_{i_2}}{r-1}$,...(infinitely many) where $A_{i_1} \supseteq A_{i_2} \supseteq ... \supseteq ....$ This implies that $A = \{x_{i_1}, x_{i_2}, ...\}$ is an infinite set such that $\binom{A}{r}$ is monochromatic. (See it ?). □

§ 6.3. **Ramsey Graph Theory**

Let $H_1, H_2, ..., H_k$ be k non-empty graphs of order $s_1, s_2, ..., s_k$ respectively. Then, we use $R(H_1, H_2, ..., H_k)$ to denote the minimum n such that for every k-coloring of $E(K_n)$, $\varphi : E(K_n) \to \{1, 2, ..., k\}$, there exists a monochromatic H_i for some $i \in \{1, 2, ..., k\}$.
(Fact) $R(H_1, H_2, ..., H_k) \leq R(s_1, s_2, ..., s_k)$.

Theorem 6.3.1. $R(sK_2, tK_2) = 2s + t - 1$. $(k = 2)$.

Proof. $(\geq)$

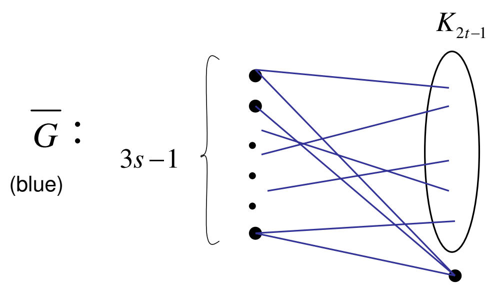


$\Rightarrow R(sK_2, tK_2) \geq 2s + t - 1$.
$(\leq)$ By induction on (s, t) $(s + t)$, $t \leq s$.
It is clear that if $t = 1$, then K_{2s} has a matching of size s. So, assume that $R((s-1)K_2, (t-1)K_2) \leq 2(s-1) + t - 2 = 2s + t - 4$ and $n = 2s + t - 1$.
First, if there is a red K_n, then clearly we have a monochromatic sK_2. Similary, if we have a blue K_n, then we have a blue tK_2 (since $s \geq t$). Therefore, there exists three vertices x, y and z such that xy is red and xz is blue. Now, consider K_{n-3} induced by $V(K_n)\backslash\{x, y, z\}$. Since $n - 3 = 2s + t - 4$, K_{n-3} contains either a red $(s-1)K_2$ or a blue $(t-1)K_2$. Thus, the proof follows by putting them together properly. □

Since $R(K_3, K_3) = 6$, we consider a more general form.

Theorem 6.3.2. $R(sK_3, tK_3) = 3s + 2t$, $s \geq t \geq 1$ and $s \geq 2$.

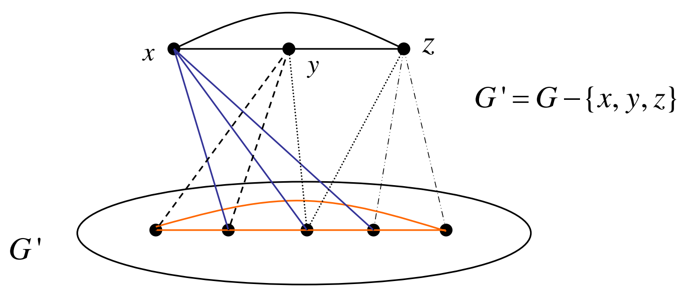
Proof. $(\geq)$ (red)$G = K_{3s-1} \cup S_{2t-1}$ (S_{2t-1} is a star with $2t-1$ edges).



Since G and $\bar{G}$ contain no sK_3 and tK_3 respectively, $R(sK_3, tK_3) \geq (3s-1) + 2t + 1 = 3s + 2t$.

$(\leq)$ First, we claim that $R(2K_3, K_3) \leq 8$.

Let $n = 8$. Consider a graph G of order 8. If $\bar{G}$ contains a triangle, then we are done. Assume that $\bar{G}$ does not contain a triangle. Therefore G contains a triangle.



Since $\bar{G}'$ contains no triangle, so is G', $G' \cong C_5$.

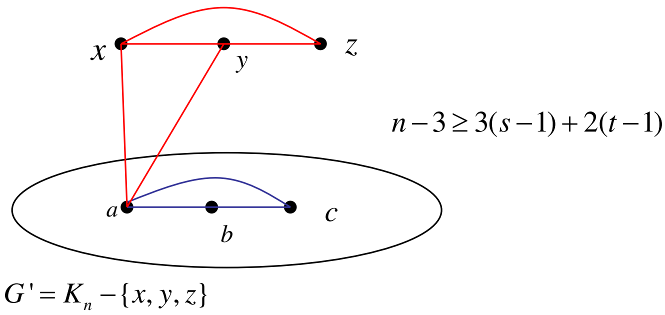
$R(2K_3, 2K_3) \leq 10$.

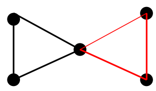
$R(sK_3, K_3) \leq 3s + 2$...By induction on (s, t).

Assume that the assertion is true for $(s-1, t-1)$, i.e.,

$R((s-1)K_3, (t-1)K_3) \leq 3(s-1) + 2(t-1)$, $s \geq t \geq 2$.

Let $n = 3s + 2t \geq 10$. In any 2-coloring φ of K_n, there exists a monochromatic triangle, say red triangle "R_3". $V(R_3) = \{x, y, z\}$. For the coloring φ on $G' = K_n - \{x, y, z\}$, we have either a red $(s-1)K_3$ or a blue $(t-1)K_3$. If the former case satisfies, then we are done. Otherwise, let B_3 be a blue triangle in G', $V(B_3) = \{a, b, c\}$. Now, consider the $K_{3,3} =< \{x, y, z\}, \{a, b, c\} >$, either there are 5 red edges or 5 blue edges. In either case, we obtain B^* (see figure below) on 5 vertices. By considering $K_n - B^*$, the proof follows.





□

Theorem 6.3.3. Let s, t be positive integers, and let T be a tree of order t. Then $R(K_s, T) = (s-1)(t-1)+1$.

Proof. $(\geq)$ Consider $G = K_{(s-1)(t-1)} \cong K_{t-1,t-1,...,t-1}$ (s-1)tuples. Then G contains no K_s and $\bar{G}$ contains no T, since $\bar{G} = \cup K_{s-1}$ $(t-1K'_{s-1}s)$. Hence $R(K_s, T) \geq (s-1)(t-1)+1$.
$(\leq)$ Let $n = (s-1)(t-1)+1$. (We claim $R(T, K_s) \leq (s-1)(t-1)+1$). Let G be a graph whose complement contains no K_s. Then $\alpha(G) \leq s-1$. This implies that $\mathcal{X}(G) \geq [(s-1)(t-1)+1]/(s-1) = t$. Since $(t \leq)\mathcal{X}(G) \leq max\{\delta(H)|H \leq G\}+1$, there exists a subgraph H of G such that $\delta(H) \geq t-1$. This implies that H contains a subgraph "T" of order $t-1$. (Note that T can be arbitrary !) □

Remarks

(1) The verticed of a tree can be enumerated (ordered), say $v_1, v_2, ..., v_n$ so that every v_i $(i \geq 2)$ has a unique neighbor in $\{v_1, v_2, ..., v_{i-1}\}$.
(2) The vertices of a connected graph G can always be enumerated, say as $v_1, v_2, ..., v_n$ so that $G_i =_{def} < \{v_1, v_2, ..., v_i\} >$ is connected for every i.