# Graph Theory I Homework III-2 by 陳哲皓

**Definition:** Let G be a graph and $x \in G$ a vertex, and let $G'$ be obtained from G by adding a vertex $x'$ and joining it to x and all the neighbours of x. We say that $G'$ is obtained from G by expanding the vertex x to an edge $xx'$

**claim1:** Any graph obtained from a perfect graph by expanding a vertex is again perfect.

*Proof.* We use induction on the order of the perfect graph considered. Expanding the vertex of $K_1$ yields $K_2$, which is perfect. For the induction step, let $G$ be a non-trivial perfect graph, and let $G'$ be obtained from $G$ by expanding a vertex $x \in G$ to an edge $xx'$ For our proof that $G'$ is perfect it suffices to show $\chi(G') \leq \alpha(G')$ : every proper induced subgraph $H$ of $G'$ is either isomorphic to an induced subgraph of $G$ or obtained from a proper induced subgraph of $G$ by expanding $x$; in either case, $H$ is perfect by assumption and the induction hypothesis, and can hence be coloured with $\omega(H)$ colours.

Let $\omega(G) =: \omega$; then $\omega(G') \in \{\omega, \omega+1\}$. If $\omega(G') = \omega+1$, then

$$\chi(G') \leq \chi(G) + 1 = \omega + 1 = \omega(G')$$

and we are done. So let us assume that $\omega(G') = \omega$. Then $x$ lies in no $K_\omega \subseteq G$ : together with $x'$, this would yield a $K_{\omega+1}$ in $G'$. Let us colour $G$ with $\omega$ colours. Since every $K_\omega \subseteq G$ meets the colour class $X$ of $x$ but not $x$ itself, the graph $H := G - (X \setminus \{x\})$ has clique number $\omega(H) < \omega$. Since $G$ is perfect, we may thus colour $H$ with $\omega - 1$ colours. Now $X$ is independent, so the set $(X \setminus \{x\}) \cup \{x'\} = V(G' - H)$ is also independent. We can therefore extend our $(\omega - 1)$-colouring of $H$ to an $\omega-$colouring of $G'$, showing that $\chi(G') \leq \omega = \omega(G')$ as desired □

**claim2:** A graph is perfect if and only if its complement is perfect.

*Proof.* Applying induction on $|G|$, we show that the complement $\bar{G}$ of any perfect graph $G = (V, E)$ is again perfect. For $|G| = 1$ this is trivial, so let $|G| \geq 2$ for the induction step. Let $\mathcal{K}$ denote the set of all vertex sets of complete subgraphs of $G$. Put $\alpha(G) =: \alpha$, and let $\mathcal{A}$ be the set of all independent vertex sets $A$ in $G$ with $|A| = \alpha$. Every proper induced subgraph of $\bar{G}$ is the complement of a proper induced subgraph of $G$, and is hence perfect by induction. For the perfection of $\bar{G}$ it thus suffices to prove $\chi\bar{G} \leq \omega\bar{G}(= \alpha)$. To this end, we shall find a set $K\mathcal{K}$ s.t $K \cap A \neq \emptyset$ for all $A \in \mathcal{A}$; then

$$\omega(\bar{G} - K) = \alpha(G - K) + 1 = \omega(\bar{G} - K) + 1 \leq \omega(\bar{G})$$

so by the induction hypothesis

$$\chi(\bar{G}) \leq \chi(\bar{G} - K) + 1 = \omega(\bar{G} - K) + 1 \leq \omega(\bar{G})$$

as desired. Supose there is no such $K$;thus, for every $K \in \mathcal{K}$ there exists a set $A_K \in \mathcal{A}$ with $K \cap A_K = emptyset$ Let us replace in $G$ every vertex $x$ by a complete graph $G_x$ of order

$$k(x) := |\{K \in \mathcal{K} | x \in A_K\}|$$

joining all the vertices of $G_x$ to all the vertices of $G_y$ whenever $x$ and $y$ are adjacent in $G$. The graph $G'$ thus obtained has vertex set $\cup_{x \in V} V(G_x)$, and two vertices $v \in G_x$ and $\omega \in G_y$ are adjacent in $G'$ iff. $x = y$ or $xy \in E$. Moreover, $G'$ can be obtained by repeated vertex expansion from the graph $G[\{x \in V | k(x) > 0\}]$. Being a induced subgraph of $G$, this latter graph is perfect by assumption, so $G'$ is perfect by claim 1.On particular,

$$\chi(G') \leq \omega(G'). \cdot (*1)$$

In order to obtain a contradiction $(*1)$, we now compute in turn the actual values of $\omega(G')$ and $\chi(G')$. By construction of $G'$, every maximal complete subgraph of $G'$ has the form $G'[\cup_{x\in X} G_x]$ for some $X \in \mathcal{K}$. So there exists a set $X \in \mathcal{K}$ s.t

$$\begin{aligned}\omega(G') &= \sum_{x\in X} k(x) \\ &= |\{(x,K) : x \in X, K \in \mathcal{K}, x \in A_K\}| = 0 \\ &= \sum K \in \mathcal{K}|X \cap A_K| \\ &\leq |\mathcal{K}| - 1 \cdots (*2)\end{aligned}$$

the last inequality follows from the fact that $|X \cap A_K| \leq 1$ for all $K$(since $A_K$ is independent but $G[X]$ is complete), and $|X \cap A_X| = 0$(by the choice of $A_X$). On the other hand,

$$\begin{aligned}|G'| &= \sum_{x\in V} k(x) \\ &= |\{(x,K) : x \in V, K \in \mathcal{K}, x \in A_K\}| \\ &= \sum_{K\in\mathcal{K}|A_K|} \\ &= |\mathcal{K}|\alpha\end{aligned}$$

As $\alpha(G') \leq \alpha$ by construction of $G'$, this implies

$$\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{G'}{\alpha} = |\mathcal{K}| \cdots (*3)$$

Putting $(*2)$ and $(*3)$ together we obtain

$$\chi(G') \geq |\mathcal{K}| > |\mathcal{K}| - 1 \geq \omega(G'),$$

a contradiction to $(*1)$ $\square$

# Graph Theory I Homework III-3 by 陳巧玲

3. Find a graph G which is C3-free and $\chi$(G)=6.

<sol>

I want to use Mycielski's construction to produce a graph G which is C3-free and

$\chi$(G)=6.

Then, I want to prove a theorem:

<Thm>From a k-chromatic triangle-free graph G' , Mycielski's construction to produce a k+1-chromatic triangle-free graph G..

<pf>

Let V(G')={$v_1 v_2 v_3 v_4 \cdots v_n$} and let G be a graph produced from it by Mycielski's construction. Let $u_1 u_2 \cdots u_n$ be the copies of $v_1 v_2 v_3 v_4 \cdots v_n$ with w the additional vertex. Let U={$u_1 u_2 \cdots u_n$}

By construction, U is an independent set in G. Hence, the other vertices of any triangle containing $u_i$ must belong to V(G'),so let $v_{k_1}, v_{k_1+1} \in V(G')$, s .t $v_{k_1} - v_{k_1+1} - u_i$ form a triangle. And this is also implies $v_i - v_{k_1} - v_{k_1+1}$ form a triangle in G'. This triangle can't exist in G'.

A proper k-coloring f of G' extends to a proper k+1-coloring of

G , by setting f($u_i$) = f($v_i$) and f(w)=k+1;hence $\chi$ (G) ≦

$\chi$ (G')+1.(1)

Now, we want to prove $\chi$ (G')< $\chi$ (G).

Let g be the proper p-coloring of G.

By changing the names of colors ,we may assume g(w)=p. So

g($u_i$) $\in \{1,2,...p-1\} \forall u_i \in U$ .

Then, we may use p colors to color V(G').Let A={ $v_i$|g($v_i$)=p

$v_j \in V(G')\}$

We want to change color of $v_j$ , where $v_j \in A$ s .t

g($v_j$)$\in$ {1,2,…p-1}

For g($u_i$) $\in \{1,2,...p-1\} \forall u_i \in U$ ,so we change the color of $v_j$ to

g($u_j$).

That is because $g(v_{j_1}) = p = g(v_{j_2})$ $\{v_{j_1}, v_{j_2}\} \notin E(G')$

$v_{j_1}, v_{j_2} \in A$

So, we only to check $\{v_j', v_j\} \in E(G')$ in order to conclude we

can use p-1 colors to color V(G'),where $v_j' \in G', v_j \in A$

For $\{v_j', v_j\} \in E(G')$ where $v_j' \in G', v_j \in A$ ,so $\{v_j', u_j\} \in E(G)$ which

yields g($u_j$)$\neq$g($v_j'$) .However, g($v_j$)=g($u_j$),so g($v_j$)$\neq$g($v_j'$).

This means ,the way we change color of $v_j$ ,where $v_j \in$A, can be used.

We have shown that the modified coloring of V(G’) is a proper

p-1 coloring of G’.

Hence, we can conclude $\chi$ (G’)< $\chi$ (G).

Also , by (1) $\chi$ (G)≦ $\chi$ (G’)+1,so $\chi$ (G’)= $\chi$ (G).

Due to this theorem , we can construct graph G ,where

$\chi$ (G)=6,step by step.

# Graph Theory I Homework III-4 by 游舜婷

4.Show that $\chi^{'}(K_{m(n)})=(m-1)n$ provided mn is even

Fact $K_{m(n)}$ can be decomposed into Hamiltonian cycles if

(1). n is even, or

(2). n is odd and m is odd

Sol: Since mn is even, we consider the following two cases

<case1> If m is odd

Since mn is even $\Rightarrow$ n is even and $\forall v\in K_{m(n)}$, $\deg(v)=(m-1)n$

(1). By the face we have $K_{m(n)}$ can be decomposed into $\frac{(m-1)n}{2}$ Hamiltonian cycles $C_{mn}$

(2). Since n is even, $C_{mn}$ is an even cycle $\Rightarrow \chi^{'}(C_{mn})=2$

$\Rightarrow$ $\therefore$ by (1) and (2) we have $\chi^{'}(K_{m(n)})=\frac{(m-1)n}{2}\times 2=(m-1)n$

Theorem (Konig)

If G is bipartite, then $\chi'(G)=\Delta(G)$

(i.e. If G=$K_{n,n}$, then $\chi'(G)=n$)

<case2> If m is even

$\Rightarrow K_m$ has 1-factorization

$\Rightarrow K_m$ has (m-1) 1-factors

$\therefore$ 將 $K_{m(n)}$ 中每個 partite set 看成一個點 $\Rightarrow K_m$

$\Rightarrow \because K_m$ has (m-1) 1-factors………………..(1)

$\therefore$ 將每一個 1-factors 對應到 $K_{m(n)}$ 會形成 $\frac{m}{2}$ 個 $K_{n,n}$

By Konig Theorem we have $\chi'(K_{n,n}) = n$………………(2)

(對每一個 1-factor 來說，我們都用這 n 個顏色去塗這 $\frac{m}{2}$

個 $K_{n,n}$，對不同的 1-factor 則用不同的 n 個顏色塗)

$\therefore$ by (1) (2) we have $\chi'(K_{m(n)}) = (m-1)n$

Hence by <case1> and <case2> we have $\chi'(K_{m(n)}) = (m-1)n$

Graph Theory I Homework III-5 by 林逸軒

5.Find a class if infinite graphs G
such that G is conformable and G is of type2

sol：$K_{2m,2m}$，m＞0

claim1：$K_{2m,2m}$ is conformable.

<u>Defn.</u> : If G has a vertex coloring Ψ such that

$$|\{i \mid |(\Psi^{-1}(i)| \not\equiv |V(G)| (\mathrm{mod} 2)\}| \leq \mathrm{def}(G)$$

Then G is *conformable.*

By χ($K_{2m,2m}$)=2，不難發現任一顏色都著了 2m點。
Let Ψ 爲點著色

$$|\{i \mid |(\Psi^{-1}(i)| \not\equiv |V(G)| (\mathrm{mod} 2)\}| = 0$$

又 def(G)=0 ( regular graph )
∴ $K_{2m,2m}$ is conformable

claim2：$K_{2m,2m}$ is of type2

由$\chi''(G) \leq \chi(G) + \chi'(G)$ 可得知

$$\chi''(K_{2m,2m}) \leq 2 + 2m - (i)$$

接下來證$\chi''(K_{2m,2m}) \geq 2 + 2m$

$$(\text{i.e}\ \chi''(K_{2m,2m}) > 2m + 1 = \Delta(G) + 1)$$

若 2m+1 色可著，則每個顏色出現$\frac{(2m)^2+4m}{2m+1}$次。

By鴿籠原理，至少存在一種顏色至少出現 2m+1 次。令至少出現 2m+1 次的顏色爲$C_i$

W.L.O.G，只討論$C_1$

∵$K_{2m,2m}$的最大matching數爲 2m，

∴必存在一些點著色爲$C_1$。

又有邊相鄰的兩點不能著同色，

∴點著色爲$C_1$的點在同一側。

∵多一點著$C_1$色，可著$C_1$色的邊數就少一。

(let t點著$C_1$色，找可著$C_1$色的邊數意同於

在$K_{2m-t,2m}$找max matching數 = 2m-t )

∴$C_1$最多只能出現 2m次 →←

得$\chi''(K_{2m,2m}) \geq 2m + 2 - (ii)$

By(i)(ii) $\chi''(K_{2m,2m}) = 2m + 2 = \Delta(G) + 2$ is type2

By 1,2 得$K_{2m,2m}$符合題意 #

# Graph Theory I Homework III-6 by 李光祥

6. For $t = 10,11,12$, find a graph of order 30 which can be decomposed into t Hamiltonian cycles and $K_{30} - G$ contains no Hamiltonian cycle.

(sol)

Definition：

(1) $D(\{i\},n)$ is the graph with vertex set $Z_n$ and edge set $\{\{x,y\} \mid |x-y| \in \{i, n-i\}\}$.

(2) If $I \subseteq \{1,2, ..., \lfloor n/2 \rfloor\}$, then let $D(I,n) = \cup_{i \in I} D(\{i\},n)$.

Theorem(Lasker and Auerbach, 1976)：

$K_{m(n)}$ has a hamiltonian decomposition if and only if $(m-1)n$ is even.

(In fact, $K_{m(n)}$ can be decomposed into $(m-1)n/2$ Hamiltonian cycles.)

Note：

$D(I,n)$ is connected if and only if $\gcd(I \cup \{n\}) = 1$.

1. $t = 10$

Let $G = K_{3(10)}$.

Since $(3-1) \times 10 = 20$ is even, by the above Theorem,

$K_{3(10)}$ can be decomposed into 10 Hamiltonian cycles.

$K_{30} - G = D(I,30)$, where $I = \{3,6,9,12,15\}$.

Since $\gcd(I \cup \{30\}) = 3 \neq 1$, $K_{30} - G$ is disconnected.

$\Rightarrow K_{30} - G$ contains no Hamiltonian cycle.

3. $t = 12$

Let $G = K_{5(6)}$.

Since $(5-1) \times 6 = 24$ is even, by the above Theorem,

$K_{5(6)}$ can be decomposed into 12 Hamiltonian cycles.

$K_{30} - G = D(I,30)$, where $I = \{5,10,15\}$.

Since $\gcd(I \cup \{30\}) = 5 \neq 1$, $K_{30} - G$ is disconnected.

$\Rightarrow K_{30} - G$ contains no Hamiltonian cycle.

Note：

(1) $D(I,n)$ is a Cayley graph if and only if it is connected.

(2) $D(I,n)$ is connected and 4-regular when $I = \{j-1, j\}$,

$2 \leq j \leq \lfloor (n-1)/2 \rfloor$.

Theorem(Bermond, Favaron and Maheo, 1989)：

Every connected 4-regular Cayley graph on a finite abelian group has a hamiltonian decomposition.

(In fact, it can be decomposed into two Hamiltonian cycles.)

2. $t = 11$

Let $G = D(I,30)$, where $I = \{1,2,4,5,7,8,9,10,11,13,14\}$.

We can partition I into $\{1,2\}$, $\{4,5\}$, $\{7,8\}$, $\{9,10\}$,$\{11\}$, $\{13,14\}$.

Since $D(\{1,2\},30)$, $D(\{4,5\},30)$, $D(\{7,8\},30)$, $D(\{9,10\},30)$, $D(\{13,14\},30)$ are connected 4-regular Cayley graphs, by the above Theorem, each of them can be decomposed into two Hamiltonian cycles.

Since $D(\{11\},30)$ is a connected 2-regular graph, it is a Hamiltonian cycle.

Hence G can be decomposed into 11 Hamiltonian cycles.

$K_{30} - G = D(I',30)$, where $I' = \{3,6,12,15\}$.

Since $\gcd(I' \cup \{30\}) = 3 \neq 1$, $K_{30} - G$ is disconnected.

$\Rightarrow K_{30} - G$ contains no Hamiltonian cycle. $\square$

Graph Theory I Homework III-7 by 李柏瑩

7. Give as many sufficient conditions for the existence of the Hamiltonian cycles in a graph as possible.

Sol:

Theorem1:
*Every graph with n* > 3 *vertices and minimum degree at least n*=2 *has a Hamilton cycle.* (Dirac 1952)
*proof* :

Let $G = (V;E)$ be a graph with $|G| = n > 3$ and $\delta(G) \geq \frac{n}{2}$.

Then $G$ is connected

Let $P = x_0 x_1 \cdots x_k$ be a longest path in $G$.

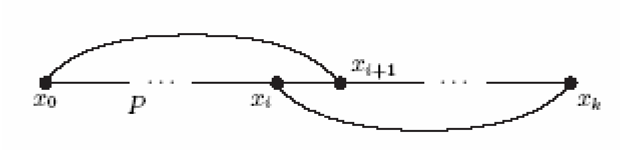
By the maximality of $P$,all the neighbors of $x_0$ and all the neighbors of $x_k$ lie on $P$.

Hence

at least $\frac{n}{2}$ of the vertices $x_0 x_1 \cdots x_{k-1}$ are adjacent to $x_k$, and at least $\frac{n}{2}$ of these same $k < n$ vertices $x_i$ are such that $x_0 x_{i+1} \in E$.

By the

Pigeonhole principle, there is a vertex $x_i$ that has both properties, so we have $x_0 x_{i+1} \in E$ and $x_i x_k \in E$ for some $i < k$

We claim that the cycle $C := x_0 x_{i+1} P x_k x_i P x_0$ is a Hamilton cycle of $G$. Indeed, since $G$ is connected, $C$ would otherwise have a neighbor in $G$-$C$, which could be combined with a spanning path of $C$ into a path longer than *P*. 


*Theorem2:*

*Every graph G with* $|G| \geq 3$ *and* $\alpha(G) \leq \kappa(G)$ *has a Hamilton cycle.*

*Theorem3:*
*Every 4-connected planar graph has a Hamilton cycle.*

# Graph Theory I Homework III-8 by 劉宜君

8. Use Petersen's Theorem to prove that $X'(G) \le 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$, when G is loopless graph.

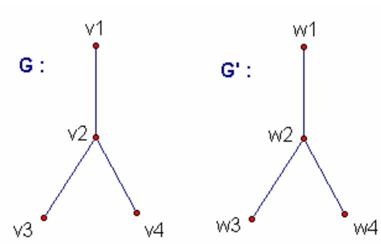
Petersen's Theorem: Every 2k-regular graph has a 2-factor.

Pf. G is a loopless graph, $V(G)=\{v_1, v_2, \ldots, v_m\}$

Let G' be a graph with $V(G')=\{w_1, w_2, \ldots, w_m\}, E(G)=\{w_i w_j \mid v_i v_j \in E(G)\}$.

Let $H = G \cup G' \cup \{\Delta(G)-\deg(v_i)$ copies of the edge $v_i w_j\}$

Then H be a $\Delta(G)$ -regular graph .



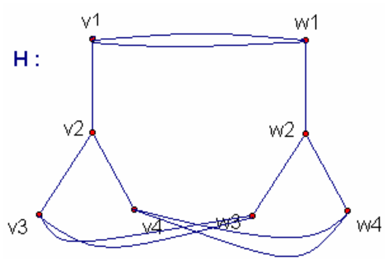

(i) If $\Delta(G)$ is even, then by Petersen's Theorem.

We can partition $H$ into $\frac{\Delta G}{2}$ 2-factors.

Since each 2-factor is disjoint union of cycles,

and $X'(C_n) \le 3$ $\left( X'(C_n) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases} \right)$

$\Rightarrow X'(H) \le 3 \times \frac{\Delta(G)}{2}$.



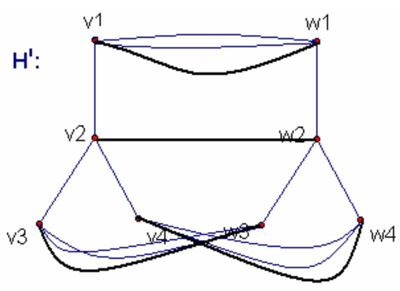

(ii) If $\Delta(G)$ is odd. Consider $H' = H \cup \{v_i w_j\}_{i=1}^{m}$

Then $H'$ be a ($\Delta(G)+1$)-regular graph and $\Delta(G)+1$ is even, by Petersen's Theorem.

We partition $H$ into $\frac{\Delta G + 1}{2}$ 2-factors.

Since each 2-factor is disjoint union of cycles,

$\Rightarrow X'(H') \le 3 \times \frac{\Delta(G)+1}{2}$.




By the definition of $H$ and $H'$, and (i) (ii) we can find that there exist a loopless supergraph $H_s$ of G , such that $X'(H_s) \le 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$.

We delete those edges of $(H_s - G)$ to obtain a proper edge-coloring of G such that

$$X'(G) \leq 3\left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

# Graph Theory I Homework III-9 by 劉士慶

9. Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$.

Consider a k-coloring graph of a k-chromatic G.

I. If $e(G) < \binom{k}{2}$, this means that there exist two colors a, b such that any adjacent vertices do not require the two colors.( $\Leftrightarrow$ the two colors would not form any edges of G). Thus the vertices with colors a and b form a single independent set, and V(G) is covered by k-1 independent sets.
A contraction, since a k chromatic graph has k independent sets.

II. Since G is the union of m complete graph of order m, which implies.

Let $\chi(G) = k$ , then $\binom{k}{2} \le m\binom{k}{2} \Rightarrow k^2 - k - m^2(m-1) \le 0$

$\because \sqrt{x+1} \le \sqrt{x} + 1, \forall x \ge 0$

$\therefore k \le \frac{1}{2}[1 + \sqrt{1 + 4m^2(m-1)}] \le [1 + (1 + \sqrt{4m^2(m-1)})] = 1 + m\sqrt{(m-1)}$

Hence $\chi(G) = k \le 1 + m\sqrt{m-1}$ .

# Graph Theory I Homework III-10 by 曾慧棻

## Graph Theory

Hui-Fen,Tseng

January 26, 2008

10.Improvement of Brooks' Theorem

(a) Given a graph G, let $k_1, ..., k_t$be nonnegative integers with $\sum k_i \geq \Delta(G) - t + 1$. Prove that V(G) can be partitioned into sets $V_1, ..., V_t$ so that for each i, the subgraph $G_i$ induced by $V_i$ has maximum degree at most $k_i$.

(b) For $4 \leq r \leq \Delta(G) + 1$, use part (a) to prove that $X(G) \leq \lceil \frac{r-1}{r}(\Delta(G) + 1) \rceil$ when G has no r-clique.

pf:

(a) By induction on t

If t=2, we claim the partition minimize $k_1 e(G_2) + k_2 e(G_1)$ satisfying $\Delta(G_1) \leq k_1$ and $\Delta(G_2) \leq k_2$

Suppose not, there exists a vertex x in $G_1$ s.t. $deg_G(x) > k_1$, it implies $deg_G(x) \geq k_1 + 1$. Since $k_1 + k_2 \geq \Delta(G) - 2 + 1 = \Delta(G) - 1$, i.e., $\Delta(G) \leq k_1 + k_2 + 1$, we obtain $|N_{G_2}(x)| \leq k_2$ in $V(G_2)$

Next, we move x in $G_1$ to $G_2$ to form $G_1' and G_2'$,
we obtain

$$\begin{aligned} k_1 e(G_2') + k_2 e(G_1') &= k_1(e(G_2) + k_2) + k_2(e(G_1) - k_1 - 1) \\ &= k_1 e(G_2) + k_1 k_2 + k_2 e(G_1) - k_2 k_1 - k_2 \\ &= k_1 e(G_2) + k_2 e(G_1) - k_2 \end{aligned}$$

we find another partition $k_1 e(G_2') + k_2 e(G_1')$ is less than $k_1 e(G_2) + k_2 e(G_1)$, it contradicts. We have done.

If $t > 2$. Let $h = k_1 + k_2 + ... + k_{t-1} + (t-2)$
Suppose $k_1, k2, ..., k_{t-1}$ is ok.
If $\Sigma_{i=1}^{t} k_i \geq \Delta(G) - t + 1$, it follow that

$k_1 + k_2 + ..., k_{t-1} + k_t \geq \Delta(G) - t + 1$
$\Rightarrow h + k_t \geq \Delta(G) - t + 1 + t - 2$
$\Rightarrow h + k_t \geq \Delta(G) - 1$
By t=2 method, we obtain $\Delta(G_h) \leq k_h, \Delta(G_t) \leq k_t$, it follows that $\Delta(G_i) \leq k_i$ for i=1,2,...,t. We have done.

(b) $r \leq \Delta(G) + 1$ it implies, there exists t s.t. $(\Sigma_{i=1}^{t-1} r) + r^* = \Delta(G) + 1$, where $2r > r^* \geq r$.Actually, $t = \lfloor \frac{\Delta(G)+1}{r} \rfloor$.
Let $k_1 = k_2 = ... = k_{t-1} = r - 1. k_t \geq r - 1$ and
$\Sigma_{i=1}^{t} = \Delta(G) - t + 1$, where $4 \leq r \leq \Delta(G) + 1$
$\Rightarrow \Delta(G) - t + 1 \geq t(r-1) = tr - t$
$\Rightarrow t \leq \frac{\Delta(G)+1}{r}$
$\Rightarrow t \leq \lfloor \frac{\Delta(G)+1}{r} \rfloor$
By (a) V(G) can partition into t classes s.t. $\Delta(G_i) \leq k_i$,i=1,...,t. Since G has no r-clique, $G_i$ has no r-clique.i=1,...,t. Thus,by Brooks' theorem, we have $X(G_i) \leq \Delta(G_i)$,it follows that $X(G_i) \leq k_i$ for all i=1,...,t. Now, coloring the subgraphs with disjoint color sets, we have

$$\begin{aligned} X(G_i) \leq \Sigma_{i=1}^{t} X(G_i) \leq \Sigma_{i=1}^{t} k_t &= \Delta(G) + 1 - t \\ &\leq \Delta(G) + 1 - \lfloor \frac{\Delta(G)+1}{r} \rfloor \\ &\leq \lceil \frac{r-1}{r} (\Delta(G) + 1) \rceil. \end{aligned}$$