Graph Theory I Homework II-1 by 陳巧玲

（由於突發狀況，臨時用了掃描版本。 by TA）

NO: 96225?6
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1. Prove the A-B version of Menger's Theorem

We denote by $k = k(G, A, B)$ the minimum number of vertices separating A from B in G. Clearly, G cannot contain more than k disjoint A-B paths; our will be show that k such paths exist.

<First proof>

We prove the following stroger statement:

If $\mathcal{P}$ is any set of fewer than k disjoint A-B paths in G then there is a set $\mathcal{Q}$ of $|\mathcal{P}|+1$ disjoint A-B paths whose set of ends includes the set of ends of the paths in $\mathcal{P}$.

Keeping G and A fixed, we let B vary and apply induction on $|G-B|$. Let R be an A-B path that avoids the vertices (fewer than k) of B that lie on a path in $\mathcal{P}$. If R avoids all the paths in $\mathcal{P}$, then $\mathcal{Q} := \mathcal{P} \cup \{R\}$ is as desired. (This will happen for $|G-B| = 0$ when all A-B paths are trivial.) ✓

If not, let x be the last vertex of R that lies on some $P_1 \in \mathcal{P}$. Put $B' := B \cup V(xP_1 \cup xR)$ and $\mathcal{P}' := (\mathcal{P} \setminus \{P_1\}) \cup \{P_1x\}$.

Then $|\mathcal{P}'| = |\mathcal{P}|$ and $k(G, A, B') \geq k(G, A, B)$, so by the induction hypothesis there is a set $\mathcal{Q}'$ of $|\mathcal{P}|+1$ disjoint A-B' paths whose ends include those of the paths in $\mathcal{P}'$. (for $k(G,A,B') \geq k(G,A,B) > |\mathcal{P}| > |\mathcal{P}'|$, and by the way $|B'| > |B| \Rightarrow |G-B'| < |G-B|$)

Then $\mathcal{Q}'$ contains a path Q_1 ending in x, and a unique path Q_2 whose last vertex y is not among the last vertices of the paths in $\mathcal{P}'$. If $y \notin xP_1$, we let

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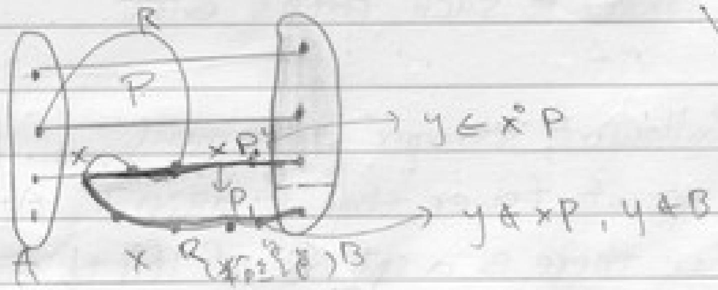
Q be obtained from Q' by adding xP_1 to Q_1 and adding yR to Q_2 if $y \notin B$. Otherwise $y \in \mathring{x}P$ and we let Q be obtained from Q' by adding xR to Q_1 and adding yP_1 to Q_2.



Paths in the first proof

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<Second proof>

We show by induction on $|G| + \|G\|$ that G contains k disjoint A–B paths. For all G, A, B with $k \in \{0, 1\}$ this is true. For the induction step let G, A, B with $k \geq 2$ be given, and assume that the assertion holds for graphs with fewer vertices or edges.

If there is a vertex $x \in A \cap B$, then $G - x$ contains $k-1$ disjoint A–B paths by the induction hypothesis. Together with the trivial path $\{x\}$, these form the desired paths in G. We shall therefore assume that $A \cap B = \emptyset$ (1) ✓

We first construct the desired paths for the case that A and B are separated by a set $X \subseteq V$ with $|X| = k$ and $X \neq A, B$. Let C_A be the union of all the components of $G - X$ meeting A; note that $C_A \neq \emptyset$, since $|A| \geq k = |X|$ but $A \neq X$. The subgraph C_B defined likewise is not empty either, and $C_A \cap C_B = \emptyset$.

Let us write $G_A := G[V(C_A) \sqcup X]$ and $G_B := G[V(C_B) \sqcup X]$. Since every A–B path in G contains an A–X path in G_A, we cannot separate A from X in G_A by fewer than k vertices.

Thus, by the induction hypothesis, G_A contains k disjoint A–X paths (Figure 1). In the same way there are k disjoint X–B paths in G_B. As $|X| = k$ we can put these paths together to form k disjoint A–B paths.

✓

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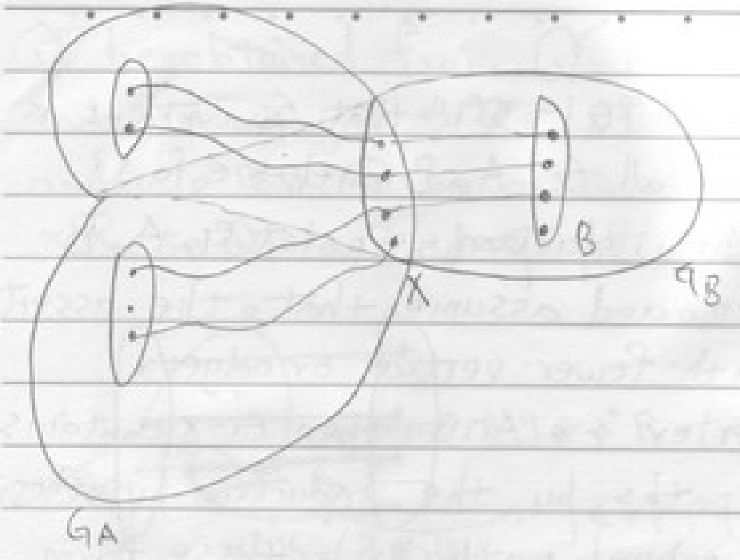
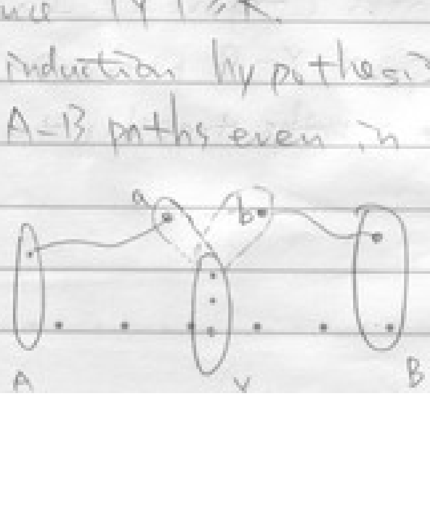


For the general case, let P be any A–B path in G.
By (1), P has an edge ab with $a \notin B$ and $b \notin A$.
Let Y be a set of as few vertices as possible separating A from B in $G - ab$ (Fig. 2). Then $Y_a := Y \cup \{a\}$ and $Y_b := Y \cup \{b\}$ both separate A from B in G, and by definition of k we have $|Y_a|, |Y_b| \geq k$. ✓
If equality holds here, we may assume by the case already threated that $\{Y_a, Y_b\} \subseteq \{A, B\}$
so $\{Y_a, Y_b\} = \{A, B\}$ since $a \notin B$ and $b \notin A$.
Thus, $Y = A \cup B$. Since $|Y| \geq k-1 \geq 1$, this contradicts (1) ✓
We therefore have either $|Y_a| > k$ or $|Y_b| > k$ and hence $|Y| \geq k$.
By the induction hypothesis, then, there are k disjoint A–B paths even in $G - ab \subseteq G$ ✓

Fig 2


separating A from B in $G - ab$

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Graph Theory I Homework II-5 by 李光祥

2. Prove the a-b version of Menger's Theorem.

Menger's Theorem(a-b version)：

The maximum number of vertex-disjoint paths between a and b is equal to the minimum number of vertices that separates a and b.

pf：

Conder $ab \notin E(G)$.

Note：If there exist k vertex-disjoint paths between a and b, then for each separating set S of a and b, $|S| \geq k$.

First, if a and b are in distinct components of G, then the Assertion is clearly true.

So let the graph G we consider be connected.

We claim that the number of vertex-disjoint paths between a and b is equal to the minimum number of vertices that separates a from b. (Let this number be n.)

Clearly, $n = 1$ is true.

Let $S_n(a,b)$ denote the statement that the minimum a-b separator is of size n.

Assume that our claim is not true.

Then there is a smallest integer m(≥ 2) such that $S_m(a,b)$

is true but there are fewer than m disjoint a-b paths.

Among all such counterexamples G of minimum order, let

H be the one with minimum size.

Then H has the following properties：

(1) If $v_1v_2 \in E(H-\{a,b\})$, then there exists a set U such

that $|U| = m-1$ and $U \cup \{v_i\}$ is an a-b separator for

$i = 1,2$.

(2) For any $w \notin \{a,b\}$ in H, not both aw and bw in H.

(3) If $\{w_1, w_2, \ldots, w_m\}$ is an a-b separator, then either

$\{aw_i \mid i = 1,2,\ldots,m\} \subseteq E(H)$ or $\{bw_i \mid i = 1,2,\ldots,m\} \subseteq E(H)$.

Let P be a shortest a-b path in H.

By (2), the length of P is at least 3.

Let $P = \langle a, u_1, u_2, \ldots, b \rangle$, where $u_1, u_2 \neq b$, $u_1 \neq u_2$.

By (1), $\exists$ U of size $m-1$ s.t $U \cup \{u_i\}$ s an a-b separator for

$i = 1,2$.

Since $au_1 \in E(H)$ but $bu_1 \notin E(H)$ (by (2)), $U \cup \{u_1\} \subseteq N_H(a)$ (by (3)).

By (2), $\forall$ x $\in$ U, xb $\notin$ E(H).

$\Rightarrow$ $U \cup \{u_2\} \subseteq N_H(a)$ (by (3)).

$\Rightarrow$ $au_2 \in E(H)$.

$\Rightarrow$ P is not a shortest a-b path in H. ($\rightarrow\leftarrow$)

(Another method)

Menger's Theorem(a-b version)：

If a,b are vertices of a graph G and ab $\notin$ E(G), then the minimum size of an a,b-cut equals the maximum number of pairwise internally disjoint a,b-paths.

pf：

Let the minimum size of an a,b-cut be κ(a,b) and the maximum number of pairwise internally disjoint a,b-paths be λ(a,b).

Since an a,b-cut must contain an internal vertex from each path in a set of pariwise internally disjoint a,b-paths and these vertices must be distinct, so $\kappa(a,b) \geq \lambda(a,b)$.

Let $k = \kappa_G(a,b)$.

Our goal：To find k pairwise internally disjoint a,b-paths.

By induction on $|G|$.

(1) $|G| = 2$

Since $ab \notin E(G)$, $\kappa(a,b) = \lambda(a,b) = 0$.

(2) $|G| > 2$

Case 1. $\exists$ a minimum a,b-cut S, $S \neq N_G(a), N_G(b)$.

Let V_1 be the set of vertices on a,S-paths, and

let V_2 be the set of vertices on S,b-paths.

Claim：$S = V_1 \cap V_2$.

Since S is a minimal a,b-cut, every vertex of S lies

on an a,b-path.

$\Rightarrow S \subseteq V_1 \cap V_2$.

If $x \in (V_1 \cap V_2) - S$, then there exists a a-…-x-…-b path

and all vertices of the path don't belong to S. ($\rightarrow\leftarrow$)

Hence $S = V_1 \cap V_2$.

Let $a' \in V_1 \setminus S$ and $b' \in V_2 \setminus S$.

Define H_1 = a,S-paths $\cup \{sb' \mid s \in S\}$, and

H_2 = S,b-paths $\cup \{a's \mid s \in S\}$.

Since every a,b'-cut in H_1 is an a,b-cut in G, we have

$\kappa_{H1}(a,b') = k$.

Similarly, $\kappa_{H2}(a',b) = k$.

Since $S \neq N_G(a), N_G(b)$, $|H_1|,|H_2| < |G|$.

By induction hypothesis, $\lambda_{H1}(a,b') = k = \lambda_{H2}(a',b)$.

Since $V_1 \cap V_2 = S$, deleting b' from the k a-b' paths in H_1 and deleting a' from the k a'-b paths in H_2 yield the desired a,S-paths and S,b-paths in G.

Case 2. Every minimum a,b-cut S is $N_G(a)$ or $N_G(b)$.

(1) If there exists a vertex $v \in V(G) \setminus (N_G[a]$ or $N_G[b])$,

then $v \notin S$.

$\Rightarrow \kappa_{G-v}(a,b) = k$.

By induction hypothesis, $\lambda_{G-v}(a,b) = k$.

$\Rightarrow \lambda_G(a,b) = k$.

(2) If there exists a vertex $u \in N_G(a) \cap N_G(b)$, then $u \in S$.

$\kappa_{G-v}(a,b) = k-1$.

By induction hypothesis, $\lambda_{G-v}(a,b) = k-1$.

$\Rightarrow \lambda_G(a,b) = k$.

(3) Assume that $G = N_G[a] \cup N_G[b]$.

Let $G' = (N_G(a),N_G(b))$ and $E(G') = [N_G(a),N_G(b)]$.

Since every a,b-path in G uses some edges from $N_G(a)$

to $N_G(b)$, the a,b-cuts in G are the vertex cover of G'.

$\Rightarrow \beta(G') = k$.

By König's Theorem, $\alpha'(G') = k$.

$\Rightarrow \lambda_G(a,b) = k$.

Then we complete the proof. □

Graph Theory I Homework II-4 by 陳哲皓

4. Prove that for almost all graphs G, $\delta(G) = k_1(G)$

Proof. .

(1) By theorem: for almost all graphs G, $diam = 2$

(2) claim: If $diam(G) = 2$, then $\delta(G) = k_1(G)$
$< pf >$
Let $d(S) = \mid [S, \bar{S}] \mid, \forall S \subseteq V(G)$ Let $\mid [S, \bar{S}] \mid$ be a mini. edge cut with $\mid S \mid \leq \mid \bar{(S)} \mid$
$\Rightarrow \mid [S, \bar{S}] \mid = k_1(G) \leq \delta(G)$

(a.) claim: every vertex of S has neighbor in $\bar{S}$
$< pf >$
assume $\exists v_0 \in S$ with $N(v_0) \cap \bar{S} = \emptyset$
$\Rightarrow deg(v_0) = d(v_0) \leq \mid S \mid -1$
$\because \mid S \mid \leq \mid \bar{S} \mid$ and $\mid S \mid + \mid \bar{S} \mid = n$
$\therefore \mid S \mid \leq \frac{n}{2} \Rightarrow deg(v_0) \leq \mid S \mid -1 \leq \frac{n}{2} - 1 < \frac{n}{2}$
$\Rightarrow \delta(G) < \frac{n}{2} \ldots (*1)$
$\because diam(G) = 2$
$\therefore$ every vertex of $\bar{S}$ has a neighbor in S (if $\exists u_0 \in \bar{S}$ with $N(u_0) \cap S$, then $d(v_0, u_0) > 2 \rightarrow\leftarrow$)
$\Rightarrow \frac{n}{2} \leq \mid \bar{S} \mid \leq \mid [S, \bar{S}] \mid = k_1(G)$
$\Rightarrow by(*1)\ \delta(G) < \frac{n}{2} \leq k_1(G)$
$\Rightarrow \delta(G) < k$
but we know $\delta(G) \geq k_1(G) \rightarrow\leftarrow$

(b.) claim: $\delta(G) = k_1(G)$
$< pf >$
by (a) $\Rightarrow \mid S \mid \leq \mid [S, \bar{S}] \mid = k_1(G) \leq \delta(G) \ldots$ (*2)
$\because \mid S \mid \leq \delta(G)$

$\therefore \forall v \in S, \mid N(v) \cap \bar{S} \mid \geq \delta(G) - (\mid S \mid -1) =$
$\delta(G) - \mid S \mid +1$
$\Rightarrow k_1(G) \geq \mid S \mid (\delta(G) - \mid S \mid +1)$
$\because k_1(G) \leq \delta(G)$
$\therefore \delta(G) \geq k_1(G) \geq \mid S \mid (\delta(G) - \mid S \mid +1)$
$\Rightarrow (\mid S \mid -1)(\delta(G) - S) \leq 0$
$\Rightarrow \mid S \mid = 1$ or $\mid S \mid \geq \delta(G)$
$\because$ by (*2) $\Rightarrow \mid S \mid \leq \delta(G)$
$\therefore \mid S \mid = 1$ or $\mid S \mid = \delta(G)$

Case 1: If $\mid S \mid = \delta(G)$

$\because \mid S \mid = \delta(G)$ and $k_1(G) \leq \delta(G)$
$\therefore k_1(G) \leq \mid S \mid$
By (a) and $k_1(G) \leq \mid S \mid$
$\Rightarrow k_1(G) = \mid S \mid$
$\because \mid S \mid = \delta(G)$
$\therefore k_1(G) = \delta(G)$

Case 2: If $\mid S \mid = 1$

$\because \mid S \mid = 1$,say $S = \{v\}$
$\therefore k_1(G) \geq deg(v) \geq \delta(G)$
we know $k_1(G) \leq \delta(G)$
$\Rightarrow k_1(G) = \delta(G)$

□

Graph Theory I Homework II-5 by 游舜婷

5. $Let\ G\ be\ an\ n\text{-}connected\ graph\ where\ n \geq 2.$
$prove\ that\ the\ join\ of\ G\ and\ K_1\ is\ (n+1)\text{-}connected.$

$pf:\ Let\ H = G \vee K_1\ and\ V(K_1) = \{v\}$

$Suppose\ H\ is\ not\ (n+1)\text{-}connected$

$Let\ S \subseteq V(H)\ be\ a\ cut\ set\ with\ |S| \leq n$

$<case1>\ If\ v \in S$

$\sin ce\ H = G \vee K_1$

$\Rightarrow\ \forall u \in V(H - (S \cup \{v\})),\ u\ is\ adjacent\ to\ v$

$\Rightarrow\ H - S\ is\ connected \quad (\rightarrow\leftarrow)$

$<case2>\ If\ v \notin S$

$Let\ S = \{v\} \dot{\cup} S'$

$Since\ G\ is\ n\text{-}connected\ and\ |S| \leq n$

$\Rightarrow\ |S'| \leq n-1$

$\therefore G - S'\ is\ connected ....(1)$

$Consider\ H - S = [H - \{v\}] - S' = G - S'$

$By\ (1)\ we\ have\ H - S\ is\ connected \quad (\rightarrow\leftarrow)$

$Hence\ by\ <case1>\ and\ <case2>\ we\ have$

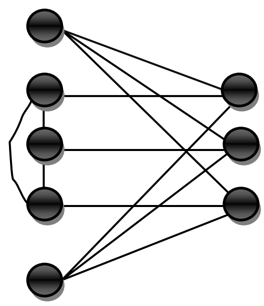
$H = G \vee K_1\ is\ (n+1)\text{-}connected$

6. Let G 3 - regular graph which is of diameter 2 and $\kappa(G)=3$.
Prove or disprove that $|E(G)|\leqq 10$.

Sol: disprove 找到反例

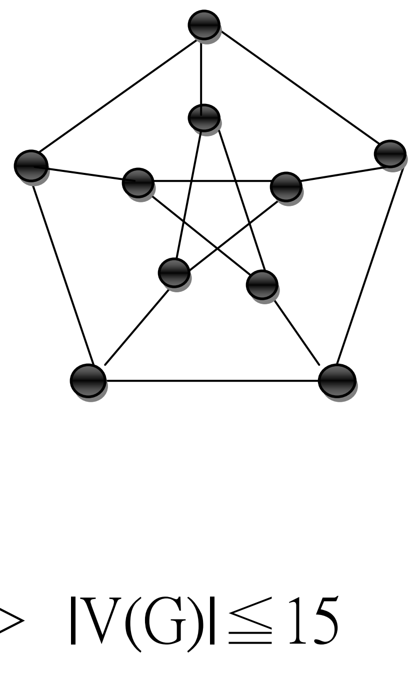
G_1: $|V(G_1)|=8$

$|E(G_1)|=12$



G_2: $|V(G_2)|=10$

$|E(G_2)|=15$



延伸證明$|E(G)|\leqq 10$ ＝＞ $|V(G)|\leqq 15$

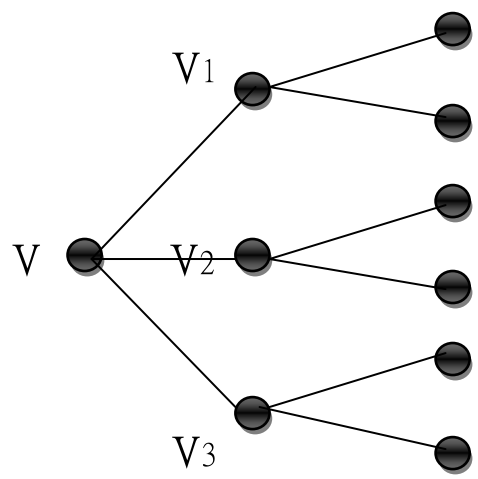
Prove: Suppose not $|V(G)|＞10$。

∵G is 3－regular ∴|V(G)|必為偶數。

現在對點v做討論，∵diameter 2 ∴若存在 2 點u,v沒有邊相連，則必存在一點v_i使得u～$_G$ v_i and v～$_G$ v_i。

$\mathbf{v_i \in N(v), i = 1,2,3}$.

但|N(N(v))\{v}|≦6，(如圖)



當|V(G)|≧12，則必存在一點 k，

使得 d(k,v)＞2。

(|{v}|+|N(v)|+ |N(N(v))\{v}|≦1+3+6＝10)

Graph Theory I Homework II-7 by 陳巧玲

7. Find as many classes of G as possible in which k(G)=k1(G).

<sol>

I find some classes:

<Thm1>If G is a simple graph with $\triangle(G) \le 3$,then k(G)=k1(G).

<pf>

For k(G)$\le$k1(G),

Now, we want to prove k(G)$\le$k1(G).

Let S is the vertex cut that $|S|$=k(G) and $v_i \in S$.

Let $H_1 = H_2$ are components of G-S

For deg(v_i)$\le$3,each v_i ahs neighbors in H_i at most 2.

We have some cases to discuss:

(1) v_i has two neighbors in H_1, this implies v_i has only one neighbor u_1 in H_2

Then cut the edge $u_1 v_1$

(2) v_i has two neighbors in H_2 , this implies v_i has only one neighbor a in H_1.

Then cut the edge $a_1 v_i$.

(3) The last case is that $v_i v_j \in E(G)$ v_i and $v_j \in S$

$$\deg_{H_1}(v_i)=1=\deg_{H_2}(v_i) \quad \deg_{H_1}(v_j)=1=\deg_{H_2}(v_j)$$

Then cut the edges to H_1 for each v_i

By (1) (2) (3), we can conclude that edges we cut is equal to $|S|$=k(G).

And these edges can separate G into many components. Hence , k(G)=k1(G).

So, W=$\{G \mid G \text{ is simple } \triangle(G) \le 3\}$ is a class.

<Thm2> k1($K_{m,n}$)=min｛m,n｝.

<pf>

Let X and Y are partite sets .

｜X｜=m ｜Y｜=n and an edge cut is an edge set of the form $[S,\bar{S}]$,where

｜$S\cap X$｜=a and ｜$S\cap Y$｜=b $1\le a+b\le m+n-1$

Then ｜$[S,\bar{S}]$｜=a(n-b)+b(m-a) the minimize:

If b=0 ,take a=｜$[S,\bar{S}]$｜=n

If b>0 ｜$[S,\bar{S}]$｜=a(n-2b)+bm

Take b=1 a=0 $\rightarrow$｜$[S,\bar{S}]$｜=m

Hence, k1($K_{m,n}$)=min｛m ,n｝

By the way, k($K_{m,n}$)=min｛m ,n｝

So, k1($K_{m,n}$)=min｛m ,n｝=k($K_{m,n}$)

<Thm3>

If G is simple and δ(G)$\ge n(G)-2$,then k(G)= δ(G).

<pf>

(1)

Let δ(G)=n-1 ,then G=K_n

So, k(G)=n-1=δ(G).

(2)

If δ(G)=n-2

Let deg(v_1)=δ(G)=n-2

This means we have $v_i\in V(G)$

s.t $v_1v_i\in E(G)$ i=2,3⋯.n-1 $v_1v_n\notin E(G)$

For $\deg(v_i) \geq n-2$ i=2,3⋯.n-1

This implies the vertices adjacent to v_i are either v_j j=2,3⋯.n-1 $i \neq j$ or v_n

Moreover, $\deg(v_n) \geq n-2$ and $v_1 v_n \notin E(G)$,this implies $v_1 v_i \in E(G)$ i=2,3⋯.n-1

i.e v_i and v_n have common neighbors v_k k=2,3⋯.n-1

So, $v_1 v_i v_n$ i=2,3⋯.n-1 are vertex-disjoint paths and the numbers of these paths are maximum. Hence, by Menger' s theorem the maximum number of vertex-disjoint paths n-2 are equal to the minimum number of vertices that separates v_i and v_n.

So, $k(G) \geq n(G)-2 = \delta(G)$.

By (1),(2) ,we can conclude that if G is simple and $\delta(G) \geq n(G)-2$,then $k(G)= \delta(G)$.

By the way, $k(G) \leqq k1(G) \leqq \delta(G)$.

So,$k(G)=k1(G)$.

So, we can construct another class $R=\{ G \mid G \text{ is simple and } \delta(G) \geq n(G)-2 \}$.

Graph Theory I Homework II-8 by 曾慧棻

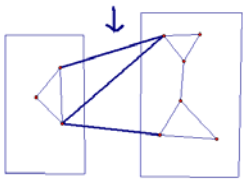
Problem 8. Show that Theorem 4.2.23 is best possible by constructing for each k, a k-connected graph having $k+1$ vertices that do not lie on a cycle.

Proof. We take $K_{k,k+1}$ to discuss. Since $K_{k,k+1}$ is k-connected with $k \geq 2$, and we can find any k vertices in a cycle. If we pick k+1 vertices in $(k+1)$-vertex part, we can't find a cycle including them, i.e.,there exists k+1 vertices that don't lie on a cycle. We have done. ∎

Graph Theory I Homework II-9 by 劉宜君

9.For $S \subseteq V(G)$, let $d(S)=\left|[S,\overline{S}]\right|$. Let X and Y be nonempty proper vertex subsrt of G.. Prove that $d(X\cap Y)+d(X\cup Y)\le d(X)+d(Y)$.

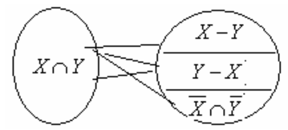
$d(S)=3$



S $\overline{S}$

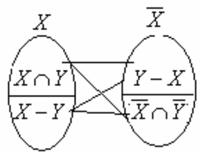
Pf. $\overline{X\cap Y}=V(G)-\{X\cap Y\}=\{X-Y\}\cup\{Y-X\}\cup\{\overline{X}\cap\overline{Y}\}$

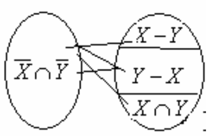
$$d\left(X\cap Y\right)=d\left[X\cap Y,X-Y\right]+d\left[X\cap Y,Y-X\right]+d\left[X\cap Y,\overline{X}\cap\overline{Y}\right]$$



$\overline{X\cup Y}=\overline{X}\cap\overline{Y}$

$$d\left(X\cup Y\right)=d\left[\overline{X}\cap\overline{Y},X-Y\right]+d\left[\overline{X}\cap\overline{Y},Y-X\right]+d\left[\overline{X}\cap\overline{Y},X\cap Y\right]$$





$\overline{X}=(Y-X)\cup(\overline{X}\cap\overline{Y})$

$$d(X)=d\left[Y-X,X\right]+d\left[\overline{X}\cap\overline{Y},X\right]$$
$$=d\left[Y-X,X\cap Y\right]+d\left[Y-X,X-Y\right]+d\left[\overline{X}\cap\overline{Y},X\cap Y\right]+d\left[\overline{X}\cap\overline{Y},X-Y\right]$$

$$d(Y)=d\left[X-Y,Y\right]+d\left[\overline{X}\cap\overline{Y},Y\right]$$
$$=d\left[X-Y,X\cap Y\right]+d\left[X-Y,Y-X\right]+d\left[\overline{X}\cap\overline{Y},X\cap Y\right]+d\left[\overline{X}\cup\overline{Y},Y-X\right]$$

$$\Rightarrow d\left(X\cup Y\right)+d(X\cap Y)=d(X)+d(Y)+2d\left[X-Y,Y-X\right]$$

$$\Rightarrow d\left(X\cup Y\right)+d(X\cap Y)\le d(X)+d(Y)$$

"=" when $d\left[X-Y,Y-X\right]=0$

Graph Theory I, Homework II-10, by 李柏瑩

Mader's theorem:

If z is a vertex of a graph G such that $\deg_G(z) \notin \{0,1,3\}$ & z is incident to no cut –edge, then z has neighbors x and y such that

$\kappa_{G-xz-yz+xy}(u,v) = \kappa_G(u,v) \quad \forall u,v \in V(G)-\{z\}$

Nash-Williams' Orientation Theorem:

Every 2k-edge-connected graph has a k edge connected orientation.
Proof: Let G be a 2k-edge-connected graph

By induction on $|G|$

If $|G|$ =2

Let V(G)={x,y}
$\Rightarrow$ x,y joined by at least 2kedges (since G is 2k-edge-connected)
$\Rightarrow$ orient at least k in each direction.

If $|G|>2$

我們透過扣邊的過程得到新的圖 G_1 such that G_1 is minimally 2k-edge-connected graph.

$\Rightarrow \delta(G_1) = 2k$ (By Exercise 4.2.37)

i.e. $z \in G_1$ such that $\deg_{G_1}(z) \notin \{0,1,3\}$

By Mader's theorem

$\Rightarrow \exists$ distinct x,y $\in$ $|G|$

such that $\kappa'_{G_1-xz-yz+xy}(u,v) = \kappa'_{G_1}(u,v)\ \forall\ u,v \in V(G_1)-\{z\}$

Let $G_2 = G_1$-xz-yz+xy

$\Rightarrow G_2$ is 2k-edge connected and $\deg_{G_2}$ (z)=2k-2

By Mader's theorem

$\Rightarrow \exists$ distinct $x', y' \in \forall u, v \in V(G_2) - \{z\}$

such that $\kappa'_{G_2 - x'z - y'z + x'y'}$(u,v)=$\kappa'_{G_2}$(u,v) $\forall$ u,v$\in$V(G_2)-{z}

Continue this process,we will obtain G_{k+1} is 2k-edge-connected and

$\deg_{G_{k+1}}(z) = 0$

Now, we consider G' = (V (G_{k+1})\{z}, E (G_{k+1}))

$\Rightarrow |G'| = |G| - 1 < |G|$

By the induction hypothesis, G' has a k-edge connected orientation.

Orient G by replacing each shortcut edge uv with the path u,z,v or v,z,u ,

Oriented consistently with uv in G'

For X$\neq$ {z},lifting uv preserves d(X)=$\left|\left[X, \overline{X} \right]\right| \geq k$ in the orientation; the

only edge lost is uv , and if uv leaves X , then uz or zv is a new edge
leaving X, depending on whether z$\in$X
The set X={z} itself reaches d(X)=k after all k lifts.