Combinatorial designs

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1.Let (**X**,**B**) be a $2-(v,b,r,k,\lambda)$ design. Prove that

(a) both $\frac{\lambda v(v-1)}{k-1}$ and $\frac{\lambda v(v-1)}{k(k-1)}$ are positive intager ;

(b) bk=vr and $r(k-1)=\lambda(v-1)$; and

(c) $b \geq v$

pf:

we use two way counting

$\mathbf{X} \to \mathbf{B}$: there are v objects and they ocuur r times, total objects is vr

$\mathbf{B} \to \mathbf{X}$: there are b blocks and their sizes are k, it implies bk. Thus, vr=bk. Done.

Fixed a point y,

$\mathbf{X} \to \mathbf{B}$: fixed y, there are v-1 objects, and each pair of distinct objects in **X** occur together in λ blocks, thus $(v-1)\lambda$.

$\mathbf{B} \to \mathbf{X}$: fixed y, some blocks' size are k-1, and other objects appear r times, thus, r(k-1). Then $\lambda(v-1)=r(k-1)$. Done.

Since $\lambda(v-1)=r(k-1)$, i.e., r=$\frac{\lambda(v-1)}{k-1}$, because $r := x \in \mathbf{X}$ occurs "r" times. Thus r is positive integer. Done.

If we choose two objects in **X**, $\binom{v}{2}$ choices, and each pair of distinct objects in **X** occur together in λ blocks, we have $\lambda\binom{v}{2}$ choices. On other hand, if we choose two objects in **B**, $\binom{k}{2}$ choices, and we have b blocks, thus b$\binom{k}{2}$ choices. Then $\lambda\binom{v}{2}$=b$\binom{k}{2}$,i.e, b=$\frac{\lambda\binom{v}{2}}{\binom{k}{2}}=\frac{\lambda v(v-1)}{k(k-1)}$ is positive integer, since b is number of blocks. Done.

(c) Given a definition: The incidence matrix of a (v,b,r,k,λ) design is a $b \times v$ matrix $A=(a_{ij})$ defined by

$$a_{ij}=\begin{cases} 1 & \text{if the ith block contain the jth element} \\ 0 & \text{otherwise} \end{cases}$$

And we show that if A is the incidence matrix of a (v,k,λ) design then $A^TA=(r-\lambda)I+\lambda J$

where A^T denote the transpose of A , I is the $v \times v$ identity matrix and J is the $v \times v$ matrix which every entry is 1.

Since A^T is $v \times b$ matrix, A^TA is $v \times v$ matrix. The (i,j) entry of A^TA is the scalar product if the i row of A^T and j column of A.
If i=j, the number of blocks containing ith object, $a_{ii} = r$
If $i \neq j$, the number of blocks containing ith object,
and the number of blocks containing jth object, thus, a_{ij}= the number of the pair (i,j) occur together, we obtain $a_{ij} = \lambda$.

Then we obtain

$$A^TA = (r-\lambda)I + \lambda J = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \ddots & \lambda \\ & & \ddots & \ddots & \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}$$

Next, we show that in any (v, k, λ) design, $b \geq v$

Let A be the incidence matrix, we first show that $det(A^TA)$ is non-zero.

$$det(A^TA) = \det(\begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \ddots & \lambda \\ & & \ddots & \ddots & \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}) = \det(\begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda - r & r-\lambda & 0 & \cdots & 0 \\ \lambda - r & 0 & r-\lambda & \ddots & 0 \\ & & \ddots & \ddots & \\ \lambda - r & 0 & 0 & \cdots & r-\lambda \end{bmatrix})$$

$$= \det(\begin{bmatrix} r+(v-1)\lambda & \lambda & \lambda & \cdots & \lambda \\ 0 & r-\lambda & 0 & \cdots & 0 \\ 0 & 0 & r-\lambda & \ddots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & r-\lambda \end{bmatrix}) = (r+(v-1)\lambda)(r-\lambda)^{v-1}$$

Since $r(k-1) = \lambda(v-1)$, $det(A^TT) = (r+(v-1)\lambda)(r-\lambda)^{v-1}$
$=(r + r(k-1))(r-\lambda)^{v-1} = rk(r-\lambda)^{v-1}$

And since $r(k-1) = \lambda(v-1)$, we know $k < v$, we obtain $r > \lambda$.
Hence, $det(A^TA) \neq 0$.

By theorem, A^TA is $v \times v$ matrix, the $det(A^TA)$ is nonzero iff A^TA is invertible. Since A^TA is invertible, $rank(A^TA) = v$.

Since $rank(A^TA) \leq rank(A^T) = rank(A)$ and $rank(A) \leq b$ (since A has b rows), we obtain $v = rank(A^TA) \leq rank(A) \leq b$, it implies $v \leq b$. We have done.

Combinatorial Designs H.W (II) 劉宜君

2.Find the 5-truples(v, b, r, k, λ) such that (a) and (b) in 1 hold, but no 2-(v, b, r, k, λ) design exist.

By Bruck–Ryser–Chowla Theorem, we know that in a symmetric (v, k, λ)-design , if v is even, then k- λ is a square; if v is odd, then the equation $z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}} \lambda y^2$ has a soluction in integers , x, y, z not all zero.

And since $\frac{\lambda(v-1)}{k-1} = r \quad \Rightarrow \lambda(v-1) = r(k-1)$, and $b = \frac{\lambda v(v-1)}{k(k-1)}$, so we only check $bk = vr$ 、 $\lambda(v-1) = r(k-1)$ and $b \in N$

(1) Choose (v, b, r, k, λ)=(22, 22, 7, 7, 2)
Satisfy $bk = vr$ 、 $\lambda(v-1) = r(k-1)$ and $b \in N$
But $22 \in 2N$, and $k - \lambda = 5$ is not square.
$\Rightarrow$ 2-(22, 22, 7, 7, 2) design is not exist.

(2) Choose (v, b, r, k, λ)=(46, 46, 10, 10, 2)
Satisfy $bk = vr$ 、 $\lambda(v-1) = r(k-1)$ and $b \in N$
But $v = 46 \in 2N$, and $k - \lambda = 8$ is not square.
$\Rightarrow$ 2-(46, 46, 10, 10, 2) design is not exist.

(3) Choose (v, b, r, k, λ)=(34, 34, 12, 12, 4)
Satisfy $bk = vr$ 、 $\lambda(v-1) = r(k-1)$ and $b \in N$
But $v = 34 \in 2N$, and $k - \lambda = 8$ is not square.
$\Rightarrow$ 2-(34, 34, 12, 12, 4) design is not exist.

Combinatorial Designs H.W (II) 李光祥

3. For each prime power q, construct a 2-(q^2+q+1, q+1, 1) design and a 2-(q^2, q , 1) design.

(sol)

Note：

(1) A 2-(q^2+q+1, q+1, 1) design with $q \geq 2$ is called a projective plane of order q.

(2) A 2-(q^2, q , 1) design with $q \geq 2$ is called an affine plane of order q.

1. Construct a 2-(q^2, q , 1) design：

Define $\mathbf{X} = Zq \times Zq$.

For any a,b $\in$ Zq, define a block $B_{a,b} = \{(x,y) \in \mathbf{X} \mid y = ax + b\}$.

For any c $\in$ Zq, define a block $B_{\infty,c} = \{(c,y) \in \mathbf{X} \mid y \in Zq\}$.

Define $\mathbf{B} = \{B_{a,b} \mid a,b \in Zq\} \cup \{B_{\infty,c} \mid c \in Zq\}$.

Claim：($\mathbf{X}$,$\mathbf{B}$) is a 2-(q^2, q , 1) design.

Clearly, there are q^2 elements in $\mathbf{X}$ and each block contains exactly q elements.

Let $(x_1,y_1), (x_2,y_2) \in \mathbf{X}$.

(1) If $x_1 = x_2$, then the unique block containing the pair

$\{(x_1,y_1), (x_2,y_2)\}$ is B_{∞,x_1}.

(2) If $x_1 \neq x_2$, then consider

$y_1 = ax_1 + b$ ----- ●

$y_2 = ax_2 + b$ ----- ▲

Goal：Show that this system of equations has a unique solution for a and b.

● − ▲ $\Rightarrow y_1 - y_2 = a(x_1 - x_2)$.

Since $x_1 \neq x_2$, $a = (x_1 - x_2)^{-1}(y_1 - y_2)$.

$\Rightarrow b = y_1 - ax_1 = y_1 - (x_1 - x_2)^{-1}(y_1 - y_2)x_1$.

Therefore, the unique block containing the pair

$\{(x_1,y_1), (x_2,y_2)\}$ is $B_{a,b}$, where $a = (x_1 - x_2)^{-1}(y_1 - y_2)$,

$b = y_1 - (x_1 - x_2)^{-1}(y_1 - y_2)x_1$.

By (1) & (2), each pair of elements is contained in a unique block.

Then (**X**,**B**) is a 2-$(q^2, q, 1)$ design.

Note：

$\forall\, i \in Zq$, let $\pi_{i+1} = \{B_{i,b} \mid b \in Zq\}$ and $\pi_{q+1} = \{B_{\infty,c} \mid c \in Zq\}$.

Then $\pi_1, \pi_2, \ldots, \pi_{q+1}$ are q+1 parallel classes of (**X**,**B**).

2. Construct a 2-(q^2+q+1, q+1, 1) design：

By 1, there is a 2-(q^2, q , 1) design (**X**,**B**) and $\pi_1, \pi_2, \ldots, \pi_{q+1}$ are q+1 parallel classes of (**X**,**B**).

Let $v_1, v_2, \ldots, v_{q+1} \notin \mathbf{X}$ and $V = \{v_1, v_2, \ldots, v_{q+1}\}$.

Define $\mathbf{X}' = \mathbf{X} \cup V$ and $\mathbf{B}' = \{B \cup \{v_i\} \mid B \in \pi_i, 1 \leq i \leq q+1\} \cup \{V\}$.

Claim：($\mathbf{X}'$,$\mathbf{B}'$) is a 2-(q^2+q+1, q+1, 1) design.

Clearly, there are q^2+q+1 elemenys in $\mathbf{X}'$ and each block contain q+1 elements.

Let $x,y \in \mathbf{X}'$.

(1) If $x,y \in \mathbf{X}$

Since (**X**,**B**) is a 2-(q^2, q , 1) design, x and y occur in a unique block in **B**.

$\Rightarrow$ x and y occur in a unique block in $\mathbf{B}'$.

(2) WLOG, $x \in \mathbf{X}$ and $y \in V$

Let $y = v_i$, then $\{x,y\} \subseteq B \cup \{v_i\}$, where $B \in \pi_i$ and B is the unique block in π_i that contains x.

$\Rightarrow$ x and y occur in a unique block in $\mathbf{B}'$.

(3) $x,y \in V$

Let $x = v_i$ and $y = v_j$.

Then $\{x,y\} \subseteq V$.

$\Rightarrow$ x and y occur in a unique block in $\mathbf{B'}$.

By (1) , (2) & (3), each pair of elements occurs in a unique block.

Then $(\mathbf{X'},\mathbf{B'})$ is a 2-$(q^2+q+1,\ q+1,\ 1)$ design. $\square$

Combinatorial Design Homework (Ⅱ) 裴若宇

4. Use recursive constructions to prove that for each $v \equiv 1 or 3 (\mathrm{mod}\, 6)$, an STS($v$) exists.

[proof]

$v \equiv 1 or 3 (\mathrm{mod}\, 6)$

$\Rightarrow v \equiv 1 or 3 or 7 or 9 (\mathrm{mod}\, 12)$

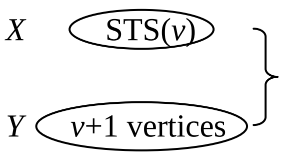
$$\Rightarrow v = \begin{cases} 12k+1 = 2(6k-3)+7 & \Rightarrow use \quad v \to 2v+7 \quad construction \\ 12k+3 = 2(6k+1)+1 & \Rightarrow use \quad v \to 2v+1 \quad construction \\ 12k+1 = 2(6k+3)+1 & \Rightarrow use \quad v \to 2v+1 \quad construction \\ 12k+1 = 2(6k+1)+7 & \Rightarrow use \quad v \to 2v+7 \quad construction \end{cases}$$

$v \to 2v+1$ construction

(Fact 1) If G is r-regular and $\chi'(G) = \Delta(G)$, then G can be decomposed into r 1-factors. (i.e. G has a 1-factorization.)

(Fact 2) K_{2m} has a 1-factorization. (可分成 $2m-1$ 個 1-factors.)

Claim If there exists an STS(v), then an STS($2v+1$) exists.



目標要把 Y 裡面還有 X, Y 之間的邊全部用完分成 K_3

Let $X = \{x_1, x_2, \cdots, x_v\}$ and $Y = \{y_1, y_2, \cdots, y_{v+1}\}$ where X and Y are disjoint.

Let (X, $\mathbf{B}_1$) be an STS(v), K_{v+1} be defined on Y.

Notice that v is odd. Thus K_{v+1} has a 1-factorization.

Let the set of 1-factors be $\{F_1, F_2, \cdots, F_v\}$

For i =1,2,…,v, use x_i and F_i to obtain $\frac{v+1}{2}$ K_3′s.

Let the collection of K_3′s obtained above be $\mathbf{B}_2$.

Notice that (1) $|\mathbf{B}_1 \cup \mathbf{B}_2| = \binom{v}{2} \cdot \frac{1}{3} + v \cdot \frac{v+1}{2} = \binom{2v+1}{2} \cdot \frac{1}{3}$,

and (2) $\forall x, y \in X \cup Y$, $\{x, y\}$ occurs in at least one K_3 of $\mathbf{B}_1 \cup \mathbf{B}_2$.

By (1) and (2), we know that $\lambda=1$.

Let $\mathbf{X} = X \cup Y$, $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2$, $(\mathbf{X}, \mathbf{B})$ is an STS$(2v+1)$.

$v \to 2v+7$ construction

(Definition) $G(D,v) =_{def} \{\{x, y\} \in E(G) \mid \|x - y\| \in D\}$ where $D \subseteq \left\{1, 2, \cdots, \left\lfloor \frac{v}{2} \right\rfloor \right\}$. (**difference** ↗ $\|x-y\|$)

(Stern & Lenz Theorem) Consider K_v. Let D be a set of differences s.t. $\exists\, d \in D$, and $\frac{v}{\gcd(d, v)}$ is even. Then $G(D,v)$ is of class 1, and thus 1-factorizable.

(Corollary) If v is even and $\frac{v}{2} \in D$, then $G(D,v)$ is 1-factorizable.

<u>Claim</u> If there exists an STS(v), then an STS($2v+7$) exists.

X (STS(v))
Y ($v+7$ vertices)

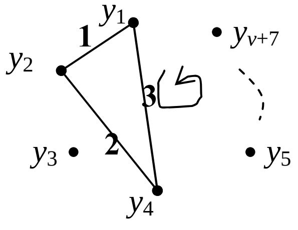
目標還是要把 Y 裡面還有 X, Y 之間的邊全部用完分成 K_3, K_{v+7} 可以分成 $v+6$ 個 1-factors, 但 X 中只有 v 點會不夠配 所以 Y 中有些邊要自己湊成 K_3.(只留下 v 個 1-factors 就好)

Let $X = \{x_1, x_2, \cdots, x_v\}$ and $Y = \{y_1, y_2, \cdots, y_{v+7}\}$ where X and Y are disjoint.

Let $(X, \mathbf{B}_1)$ be an STS(v), K_{v+7} be defined on Y.

Since $v+7$ is even, difference set $= \left\{1, 2, \cdots, \frac{v+7}{2}\right\}$

Let $G_1 = G(\{1, 2, 3\}, v+7)$. Thus G_1 can be decomposed into K_3′s. …………(*)



$K_{v+7} \backslash G_1 = G(D, v+7)$ where $D = \left\{4, 5, 6, \cdots, \frac{v+7}{2}\right\}$

Since $\frac{v+7}{2} \in D$,

by the above corollary, $G(D, v+7)$ has a 1-factorization $\{F_1, F_2, \cdots, F_v\}$.

Similarly to ***v*→2*v*+1 construction**,

for i =1,2,…,v, use x_i and F_i to obtain $\frac{v+7}{2}$ K_3′s. ···········(**)

Let the collection of K_3′s obtained by (*) and (**) be $\mathbf{B}_2$.

Notice that (1) $|\mathbf{B}_1 \cup \mathbf{B}_2| = \binom{v}{2} \cdot \frac{1}{3} + (v+7) + v \cdot \frac{v+7}{2} = \binom{2v+7}{2} \cdot \frac{1}{3}$,

and (2) $\forall x, y \in X \cup Y$, $\{x, y\}$ occurs in at least one K_3 of $\mathbf{B}_1 \cup \mathbf{B}_2$.

By (1) and (2), we know that λ=1.

Let $\mathbf{X} = X \cup Y$, $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2$, $(\mathbf{X}, \mathbf{B})$ is an STS(2v+7).

Combinatorial Designs H.W (II) 游舜婷

5. *Use direct constructions to prove that an* $STS(v)$ *exists for each* $v \equiv 1 \text{ or } 3 \pmod 6$

Definition

$Let\ H = \{H_0, H_1, H_2, ..., H_{m-1}\}$
$= \{\{0,1\},\{2,3\},\{4,5\},...,\{2m-2, 2m-1\}\}$

A latin square with holes H is latin square of order 2m which has the following properties:

(1). $\forall i, j \in H_l$, $0 \le l \le m-1$, *the* (i, j) *position of the latin square is empty* ((i, j)是空白的位置)

(2). *put the symbols belong to* Z_{2m} *in nonempty position of the latin square such that each symbol occurs exactly once in each row and column* (不是空白的位置填中的Z_{2m}元素，使得每行每列每個元素最多只出現一次)

and if $i \in H_l$, $0 \le l \le m-1$, *i doesn't appear in the ith row and the ith column*

Theorem

For all $m \ge 3$, *there exists a commutative latin square with holes* $H = \{\{0,1\},\{2,3\},\{4,5\},...,\{2m-2, 2m-1\}\}$

Pf: <Case1> 6m+1 constructions, $m \ge 3$

$m = 1,\ X = Z_7$, $B = \{\{0,1,3\} + i \pmod 7 \mid i \in Z_7\}$

$\Rightarrow STS(7)$ exists

$m = 2,\ X = Z_{13}$, $B = \{\{0,1,4\} + i, \{0,2,7\} + i \pmod{13} \mid i \in Z_{13}\}$

$\Rightarrow STS(13)$ exists

$When\ m \geq 3,\ we\ use\ 6m+1\ construction$

$Let\ X = \{\infty\} \dot{\cup} A_1 \dot{\cup} A_2 \dot{\cup} A_3$

$where\ A_1 = \{a_i \mid i \in Z_{2m}\}$

$A_2 = \{b_i \mid i \in Z_{2m}\}$

$A_3 = \{c_i \mid i \in Z_{2m}\}$

(1) $\{\infty\} \cup \{a_{2j}, a_{2j+1}\} \cup \{b_{2j}, b_{2j+1}\} \cup \{c_{2j}, c_{2j+1}\}\ ,\ \forall j \in Z_{2m}$

$\Rightarrow\ we\ can\ construct\ m\ STS(7)$

$\Rightarrow\ there\ are\ 7m\ triangles$

(2) $Since\ m \geq 3,\ by\ theorem\ there\ exists\ a\ commutative\ latin\ square$

$L\ of\ order\ 2m\ with\ holes\ H = \{\{0,1\},\{2,3\},...,\{2m-2,2m-1\}\}$

$If\ L(i,j) = k\ ,\ then\ let\ \{a_i, a_j, b_k\},\ \{b_i, b_j, c_k\},\ \{c_i, c_j, a_k\}\ be\ three$

$triangle\ in\ the\ system$

$Since\ L\ is\ of\ order\ 2m\ and\ commutative\ with\ m\ 2 \times 2\ holes$

$$there\ are\ \frac{(2m)^2 - 2^2 m}{2} * 3 = 6m^2 - 6m\ triangles$$

$Let\ the\ collection\ of\ triangles\ obtained\ above\ from\ (1)\ and\ (2)\ be\ B$

$we\ have\ |B| = 7m + 6m^2 - 6m = 6m^2 + m$

$$there\ are\ \binom{6m+1}{2} = 3m(6m+1)\ pairs$$

$By\ the\ definition\ of\ L,\ we\ can\ easily\ know\ that\ each\ pair$

$occurs\ exactly\ in\ one\ triangle\ of\ B$

$\Rightarrow \lambda = 1$

$(\because L$ *is a commutative latin square of order* $2m$ *with* m
2×2 *holes*

$\therefore$ *the triangles constructed from* (1) *and* (2) *won't have common edge*

Moreover, any two triangles constructed from (2) *won't have common edge*)

Hence (X,B) *is an* $STS(6m+1)$

<Case2> 6m+3 constructions, $m\geq 3$

$m=1,\ \mathrm{X}=Z_{13}$,

$$\mathrm{B}=\{\{0,1,2\},\{3,4,5\},\{6,7,8\}$$
$$\{0,3,6\},\{1,4,7\},\{2,5,8\}$$
$$\{0,4,8\},\{1,5,6\},\{2,3,7\}$$
$$\{0,5,7\},\{1,3,8\},\{2,4,6\}\}$$

$\Rightarrow STS(13)$ *exists*

$m=2,\ \mathrm{X}=\{1,2,3,\ldots,15\}$

$$\mathrm{B}=\{\{1,2,3\},\{4,8,12\},\{5,10,14\},\{6,11,13\},\{7,9,15\}$$
$$\{1,4,5\},\{2,8,10\},\{3,13,15\},\{6,9,14\},\{7,11,12\}$$
$$\{1,6,7\},\{2,9,11\},\{3,12,14\},\{4,10,15\},\{5,8,13\}$$
$$\{1,8,9\},\{2,12,15\},\{3,5,6\},\{4,11,14\},\{7,10,13\}$$
$$\{1,10,11\},\{2,13,14\},\{3,4,7\},\{5,9,12\},\{6,8,15\}$$
$$\{1,12,13\},\{2,4,6\},\{3,9,10\},\{5,11,15\},\{7,8,14\}$$
$$\{1,14,15\},\{2,5,7\},\{3,8,11\},\{4,9,13\},\{6,10,12\}$$

$\Rightarrow STS(15)$ *exists*

$When\ m \geq 3,\ we\ use\ 6m+3\ construction$

$Lbet\ \mathrm{X} = A_1 \cup A_2 \cup A_3$

$where\ A_1 = \{0,1,2,\ldots,2m\}$

$A_2 = \{2m+1, 2m+2, \ldots, 4m, 4m+1\}$

$A_3 = \{4m+2, 4m+3, \ldots 6m+1, 6m+2\}$

$(1)\ Let\ \mathrm{B}_i = \{i, 2m+1+i, 4m+2+i\},\ i \in \mathrm{Z}_{2m+1}$

$be\ the\ triangles\ in\ the\ system$

$\Rightarrow there\ are\ 2m+1\ triangles$

$(2)\ For\ each\ pair\ a,b \in A_i\ ,\ a \neq b\ and\ i = 1,2,3$

$define\ c \in A_{i+1(\mathrm{mod}\,3)}\ such\ that\ 2c \equiv a+b\ (\mathrm{mod}\ \ 2m+1)$

$\Rightarrow c \neq a\ (\mathrm{mod}\ \ 2m+1)\ and\ c \neq b\ (\mathrm{mod}\ \ 2m+1)$

$Let\ \{a,b,c\}\ be\ triangles\ in\ the\ system$

$Since\ c \neq a\ (\mathrm{mod}\ \ 2m+1)\ and\ c \neq b\ (\mathrm{mod}\ \ 2m+1)$

$\Rightarrow triangles\ constructed\ from\ (2)\ won't\ have\ common$

$edge\ with\ the\ triangles\ constructed\ from\ (1)$

Claim: |B|=(2m+1)(3m+1)

$Let\ the\ collection\ of\ triangles\ obtained\ above\ from$

$(1)\ and\ (2)\ be\ B$

$$then\ \ |\mathrm{B}| = 2m+1+\binom{2m+1}{2}\times 3$$

$$= (2m+1)(2m+3)$$

Claim: λ=1

Let $A_1 \cup A_2 \cup A_3$ *be an* $3\times(2m+1)$ *array*

$$\begin{bmatrix} 0 & 1 & . & . & . & . & 2m-1 & 2m \\ 2m+1 & 2m+2 & . & . & . & . & 4m & 4m+1 \\ 4m+2 & 4m+3 & . & . & . & . & 6m+1 & 6m+2 \end{bmatrix}$$

(a).*Consider the pair in the same column* i

then the pair only occurs in the triple B_i

(b).*Consider the pair in the same row*

then the pair only occurs in one triple which we constructed from (2)

(c).*Consider the pair in different row and different column*

W.L.O.G. let a *in one row and* c *in next row*

$\Rightarrow a \neq c \pmod{2m+1}$

this pair only occurs in the triple $\{a,b,c\}$ *where* b *is in the same row as* a *and* $a+b \equiv 2c \pmod{2m+1}$

by (a), (b) *and* (c) *we have* $\lambda = 1$

Thus (X,B) *is an* $STS(6m+3)$

Combinatorial Designs H.W (II)

連敏筠

#6.

If there exist three differences in $\left\{1,2,\ldots,\frac{v-1}{2}\right\}$ a, b, c such that

1. a+b=c or 2. $a+b+c \equiv 0 \pmod{v}$, then we have a starter obtained from the above difference-triples {a, b, c}. Therefore, it suffices to partition $\left\{1,2,\ldots,\frac{v-1}{2}\right\}$ into difference-triples.

Consider the following cases:

(1). 24k+1
(2). 24k+7
(3). 24k+13
(4). 24k+19
(5). 24k+3
(6). 24k+21
(7). 24k+9
(8). 24k+15

(1). 24k+1

$\langle 1,2,\ldots,12k \rangle$: set of differences

1,2,…,4k | <u>4k+1,…,12k</u>

⇓

Skolem sequence of type A

$$\{(a_1,b_1),\ldots,(a_{4k},b_{4k})\}\ s.t\ b_i - a_i = i \Rightarrow (i,a_i,b_i) : difference-triples.$$

(2). 24k+7

$\langle 1,2,\ldots,12k+3 \rangle$: set of differences

1,2,…,4k+1 | <u>4k+2,…,12k+3</u>

⇓

Skolem sequence of type A

$$\{(a_1,b_1),\ldots,(a_{4k+1},b_{4k+1})\}\ s.t\ b_i - a_i = i \Rightarrow (i,a_i,b_i) : difference-triples.$$

(3). 24k+13

$\langle 1,2,\ldots,12k+6 \rangle$: set of differences

1,2,…,4k+2 | 4k+3,…,12k+6

⇓12k+6=12k+7

Skolen sequence of type B

$$\{(a_1,b_1),\ldots,(a_{4k+2},b_{4k+2})\}\ s.t\ b_i-a_i=i\ \ and\ (a_j,12k+7)\ \ for\ some\ j$$

$$\Rightarrow 1.(i,a_i,b_i)\ if\ \ i\neq j$$

$$2.(j,a_j,12k+6): difference-triples.$$

(4). 24k+19

$\langle 1,2,\ldots,12k+9\rangle$: set of differences

1,2,…,4k+3 | 4k+4,…,12k+9

⇓12k+9=12k+10

Skolem sequence of type B

$$\{(a_1,b_1),\ldots,(a_{4k+3},b_{4k+3})\}\ s.t\ b_i-a_i=i\ \ and\ (a_j,12k+10)\ \ for\ some\ j$$

$$\Rightarrow 1.(i,a_i,b_i)\ if\ \ i\neq j$$

$$2.(j,a_j,12k+9): difference-triples.$$

(5). 24k+3

$\langle 1,2,\ldots,12k+1\rangle$: set of differences

Consider (i) a=b=c=8k+1 and

(ii) <1,2,…,12k+1>\{8k+1}

1,2,…,4k | 4k+1,…,8k,8k+2,…,12k+1

因爲{4k+1,...,8k,8k+1,8k+2,…12k+1}: difference set 可以拿掉

12k+1 s.t {4k+1,...,8k,8k+1,8k+2,…12k}: skolem sequence of type A.

所以{4k+1,...,8k,8k+2,…12k+1}是一個 skolem sequence of type A.

$$\{(a_1,b_1),\ldots,(a_{4k},b_{4k})\}\ s.t\ b_i-a_i=i$$

$$\Rightarrow 1.(i,a_i,b_i)$$

$$2.(8k+1,8k+1,8k+1): difference-triples.$$

(6). 24k+21

$\langle 1,2,\ldots,12k+10\rangle$: set of differences

Consider (i) a=b=c=8k+7 and

(ii) <1,2,…,12k+10>\{8k+7}

1,2,…,4k+3 | 4k+4,…,8k+6,8k+8,…,12k+10

因爲{4k+4,...,8k+6,8k+7,8k+8,…12k+10}: difference set 可以

拿掉 12+9 s.t {4k+4,...,12k+8,12k+10}: skolem sequence of type B.

所以{4k+4,...,8k+6,8k+8,…12k+10}是一個 skolem sequence of type B.

$$\{(a_1,b_1),...,(a_{4k+3},b_{4k+3})\}\ s.t\ b_i-a_i=i$$

$$\Rightarrow 1.(i,a_i,b_i)$$

$$2.(8k+7,8k+7,8k+7): difference-triples.$$

(7). 24k+9

$\langle 1,2,...,12k+4\rangle$: set of differences

Consider (i) a=b=c=8k+3 and

(ii) <1,2,…,12k+4>\{8k+3}

1,2,…,4k+1 | 4k+2,…,8k+2,8k+4,…,12k+4

Take the triples:

(1) (1,11k+4,11k+5)

(2) (2r+1,10k+2-r,10k+r+3) where $1\le r\le k$

(3) (2k+3,10k+2,12k+4)

(4) (2k+2r+3,9k-r+2,11k+r+5) where $1\le r\le k-2$

(5) (4k+1,6k+2,10k+3)

(6) (2r,6k+2-r,6k+2+r) where $1\le r\le 2k$

(7) (8k+3,8k+3,8k+3)

(8). 24k+15

$\langle 1,2,...,12k+7\rangle$: set of differences

Consider (i) a=b=c=8k+5 and

(ii) <1,2,…,12k+7>\{8k+5}

1,2,…,4k+2 | 4k+2,…,8k+4,8k+6,…,12k+7

Take the triples:

(1)(1,11k+5,11k+6)

(2)(2r+1,6k+3-r,6k+r+4) where $1 \le r \le 2k$
(3)(2,12k+6,12k+7)
(4)(2r,10k-r+5,10k+r+5) where $2 \le r \le k-1$
(5)(2k,8k+6,10k+6)
(6)(2k+2r,9k+6-r,11k+6+r) where $1 \le r \le k-1$
(7)(4k+2,6k+3,10k+5)
(8)(8k+5,8k+5,8k+5)

7. For each v $\equiv$ 3(mod 6) a KTS(v) exists. 施智懷

proof:

Theorem (Brouwer)

$$\forall v \equiv 1 (\text{mod } 3),\ v \notin \{10,19\},\ a\ (v,\{4,7\},1)-\textit{design exists}.$$

Fact : KTS(9) , KTS(15) exists.

(1.) KTS(9) : (X,B)

$X = \{1,2,\cdots,9\}$

$B = \{\{1,2,3\},\{4,5,6\},\{7,8,9\},$

$\{1,4,7\},\{2,5,8\},\{3,6,9\},$

$\{1,6,8\},\{2,4,9\},\{3,5,7\},$

$\{1,5,9\},\{2,6,7\},\{3,4,8\}\}$

(2.) KTS(21): (X,B)

$X = \{a,1,2,\cdots,7,11,12,13\cdots,17\}$

$B = \{\{a,1,11\},\{2,4,13\},\{3,7,15\},\{5,6,12\},\{14,16,17\},$

$\{a,2,12\},\{3,5,14\},\{4,1,16\},\{6,7,13\},\{15,17,11\},$

$\{a,3,13\},\{4,6,15\},\{5,2,17\},\{7,1,14\},\{16,11,12\},$

$\{a,4,14\},\{5,7,16\},\{6,3,11\},\{1,2,15\},\{17,12,13\},$

$\{a,5,15\},\{6,1,17\},\{7,4,12\},\{2,3,16\},\{11,13,14\},$

$\{a,6,16\},\{7,2,11\},\{1,5,13\},\{3,4,17\},\{12,14,15\},$

{a,7,17},{1,3,12},{2,6,14},{4,5,11},{13,15,16}}

$$Claim: v = 6h+3 = 2(3h+1)+1 \ \ (h \neq 3,6 \ i.e. \ 3h+1 \neq 10,19)$$
$$then \ KTS(v) \ exists$$

let $\boldsymbol{X} = \{w, u_1, u_2, ..., u_{3h+1}, v_1, v_2, ..., v_{3h+1}\}$

let $(\{u_1, ..., u_{3h+1}\}, B_1)$ *is a* $(3h+1, \{4,7\}, 1) - design$, where $B_1 = \{\mathrm{B}_1, ..., \mathrm{B}_t\}$

copy $\downarrow$ $i.e. \ def \ f: \{u_1, ..., u_{3h+1}\} \to \{v_1, ..., v_{3h+1}\} \ s.t. \ f(u_i) = f(v_i), \ \forall i \ \ f(\mathrm{B}_i) = \mathrm{B}_i^*$

let $(\{v_1, ..., v_{3h+1}\}, B_2)$ *is a* $(3h+1, \{4,7\}, 1) - design$, where $B_2 = \{\mathrm{B}_1^*, ..., \mathrm{B}_t^*\}$

$If \ |\mathrm{B}_i| = 4, \ \mathrm{B}_i = \{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}\}$

$let \ (\mathrm{B}_i \cup \mathrm{B}_i^* \cup \{w\}, \overline{B_i}\} \ is \ a \ KTS(9) \ and \ \{w, u_{i,j}, v_{i,j}\} \in \overline{B_i} \ \ \forall 1 \le i \le 4$

$If \ |\mathrm{B}_i| = 7, \ \mathrm{B}_i = \{u_{i,1}, u_{i,2}, ..., u_{i,7}\}$

$let \ (\mathrm{B}_i \cup \mathrm{B}_i^* \cup \{w\}, \overline{B_i}\} \ is \ a \ KTS(15) \ and \ \{w, u_{i,j}, v_{i,j}\} \in \overline{B_i} \ \ \forall 1 \le i \le 7$

➔ $We \ can \ partition \ \overline{B_i} \to parallel \ classes \ \overline{B_{i,1}}, ..., \overline{B_{i,4}} \ if \ |\mathrm{B}_i| = 4$

$(We \ can \ partition \ \overline{B_i} \to parallel \ classes \ \overline{B_{i,1}}, ..., \overline{B_{i,7}} \ if \ |\mathrm{B}_i| = 7 \ )$

Let $B_i^{'} = \bigcup_{\{w, u_i, v_i\} \in \overline{B_{i,j}}} \overline{B_{i,j}} \ , \ \forall 1 \le i \le 3h+1$

➔ $(\{w, u_1, u_2, ..., u_{3h+1}, v_1, v_2, ..., v_{3h+1}\}, \bigcup_{i=1}^{3h+1} B_i^{'}) \ is \ a \ KTS(v)$

Claim: KTS(21),KTS(39) exist.

Theorem. Let $q = p^{\alpha} = 6m+1$. Then a resolvable Steiner triple system of order 3q exists.

$\because$ *q is a prime power* $\Rightarrow$ *let* ρ *be a primitive* element of GF(q)

$A = \{O_1, O_2, O_3\}$

$B_{i,j} = \{\rho_j^{i}, \rho_j^{i+2m}, \rho_j^{i+4m}\}, 1 \le i \le m, 1 \le j \le 3$

$C_{i,j} = \{\rho_j^{i+m}, \rho_{j+1}^{i+3m}, \rho_{j+2}^{i+5m}\}, 1 \le i \le m, 1 \le j \le 3 (\mathrm{mod}\ 3)$

$D_{i,j} = \{\rho_j^{i}, \rho_{j+1}^{i+2m}, \rho_{j+2}^{i+4m}\}, 1 \le i \le m, 1 \le j \le 3 (\mathrm{mod}\ 3)$

The sets $A, B_{i,j}, C_{i,j} (1 \le i \le m, 1 \le j \le 3)$ form one resolution class and the translates give afuther 6m classes.

Finally each Di,j with its translates give a futher resolution class; so we obtain a futher 3m classes , giving $9m+1$ in all.

BY THEOREM ➔ KTS(21),KTS(39) EXIST!

Combinatorial Designs H.W (II) 吳政軒

8. Let $v \equiv 3 \pmod 6$. Prove that $K_V \setminus C_V$ can be decomposed into triangles.

pf : Let D be a difference set of K_V .

Case1 :

$V = 24t+3$, $D=\{1,2,\ldots,12t+1\}$

Since g.c.d(24t+3,12t+1) = 1

∴ the difference 12t+1 可造一個 C_V

hence it suffices to partition $D' =\{1,2,\ldots,12t\}$

∵ $\{4t+1,4t+2,\ldots,12t\} = 8t = 2\times4t$

∴$\{4t+1,4t+2,\ldots,12t\} = \{a_i , b_i : 1\le i \le 4t\}$

is a Skolem sequence of type A

$\forall i , b_i - a_i = i \quad 1\le i \le 4t$ and

(i , a_i , b_i) is a difference-triple of type1

hence $K_V \setminus C_V$ can be decomposed into triangles.

Case2 :

$V = 24t+9$, $D=\{1,2,\ldots,12t+4\}$

Since g.c.d(24t+9,12t+4) = 1

∴ the difference 12t+4 可造一個 C_V

hence it suffices to partition $D' =\{1,2,\ldots,12t+3\}$

∵ $\{4t+2,4t+3,\ldots,12t+3\} = 8t+2 = 2 \times (4t+1)$

∴$\{4t+2,4t+3,\ldots,12t+3\} = \{a_i , b_i : 1\le i \le 4t+1\}$

is a Skolem sequence of type A

$\forall i , b_i - a_i = i \quad 1\le i \le 4t+1$ and

(i , a_i , b_i) is a difference-triple of type1

hence $K_V \setminus C_V$ can be decomposed into triangles.

Case3 :

$V = 24t+15$, $D=\{1,2,\ldots,12t+7\}$

Since g.c.d(24t+15,6t+4) = 1

∴ the difference 6t+4 可造一個 C_V

hence it suffices to partition $D' =\{1,2,\ldots,12t+7\} \setminus \{6t+4\}$

∵ $\{4t+3,4t+4,\ldots,6t+3,6t+5,\ldots,12t+7\} = 8t +4 = 2\times (4t+2)$

∴$\{4t+3,4t+4,\ldots,6t+3,6t+5,\ldots,12t+7\} = \{a_i , b_i : 1\le i \le 4t+2\}$

is a Skolem sequence of type B

$\forall i , b_i - a_i = i \quad 1\le i \le 4t+2$ and

(i , a_i , b_i) is a difference-triple of type1

hence $K_V \setminus C_V$ can be decomposed into triangles.

Case4 :

$V = 24t+21$, $D=\{1,2,\ldots,12t+10\}$

Since g.c.d$(24t+15,6t+5) = 1$

$\therefore$ the difference $6t+5$ 可造一個 C_V

hence it suffices to partition $D' =\{1,2,\ldots,12t+10\} \setminus \{6t+5\}$

$\because$ $|\{4t+4,4t+5,\ldots,6t+4,6t+6,\ldots,12t+10\}| = 8t+6 = 2\times(4t+3)$

$\therefore \{4t+4,4t+5,\ldots,6t+4,6t+6,\ldots,12t+10\} = \{a_i , b_i : 1\le i \le 4t+3\}$

is a Skolem sequence of type B

$\forall\, i$, $b_i - a_i = i$ $1\le i \le 4t+3$ and

(i , a_i , b_i) is a difference-triple of type1

hence $K_V \setminus C_V$ can be decomposed into triangles.

10 Let $G = (V, E)$ be a graph where (X, B) is an $STS(v)$, $V = B$, and two vertices in G are adjacent if and only if they have one element in common.Find diam(G).

Proof. . Let $G = (X, B)$ be a graph where (X, B) is an $STS(v)$, $V = B$, and two vertices in G are adjacent if and only if they have one element in common.

(1) claim: $diam(G) \leq 2$
$< pf >$
If $\exists v_1, v_2 \in V(G)$ s.t $v_1v_2 \notin E(G)$, then $v_1 \cap v_2 = \emptyset$.
Let $a \in v_1, b \in v_2$
$\because v_1 \cap v_2 = \emptyset \therefore a \neq b$
$\because (X, B)$ is an $STS(v) \therefore a, b$ occurs in exactly one triangle in B
$\Rightarrow \exists$ some $v_k \in B$ s.t $\{a, b\} \subseteq v_k$
$\Rightarrow v_k \cap v_1 = \{a\}$ and $v_k \cap v_2 = \{b\}$
$\Rightarrow v_kv_1$ and $v_kv_2 \in E(G)$
$\Rightarrow d(v_1, v_2) = 2$

(2) If $v = 3$
clearly $G = K_1 \Rightarrow diam(G) = 1$

(3) If $v = 7$
$| V |= (7 \times 6)/6 = 7$
Give any $v_0 \in V$, $deg(v_0) = (\frac{7-1}{2} - 1)3 = 6$
$\because | V |= 7$ and $deg(v_0) = 6, \forall v_0 \in G. \therefore G = K_7$
$\Rightarrow diam(G) = 1$

(4) If $v \equiv 7$ or $3 \pmod 6$,$v \neq 3$ and 7
Given any $v_0 \in V(G)$,$deg(v_0) = \frac{(v-1)}{2} - 1 = \frac{3(v-3)}{2}$

(a) If $v = 6k+1$, then $| V | -deg(v_0) = \frac{(6k+1)(6k)}{6} - \frac{3(6k-2)}{2} = k(6k+1) - 3(3k-1) = 6k^2 - 8k + 3$
$\because v \neq 7 \therefore k \geq 2 \Rightarrow | V | -deg(v_0) > 1$
If $v = 6k+3$, then $| V | -deg(v_0) = \frac{(6k+3)(6k+2)}{6} - \frac{3(6k)}{2} = k(2k+1)(3k+1) - 9k = 6k^2 - 4k + 1$
$\because v \neq 3 \therefore k \geq 1 \Rightarrow | V | -deg(v_0) > 1$

By (a) $\Rightarrow | V | -deg(v_0) > 1, \forall v_0 \in V(G)$
$\Rightarrow \forall v_0 \in V(G), \exists v_1 \in V(G)$ s.t $v_1 \notin N(V_0)$

But by (1) $\Rightarrow$ If $v_1 \notin N(v_0)$ then $d(v_0, v_1) = 2$
$\Rightarrow diam(G) = 2$

(5) By (1),(2),(3),(4) $\Rightarrow$
If $v = 3$ or $7 \Rightarrow diam(G) = 1$
If $v \equiv 7$ or $3 \pmod 6$,$v \neq 3$ and $7 \Rightarrow diam(G) = 2$

□