Graph Theory I Homework I-1 by 李光祥

1. Find an infinite class of bipartite graphs G such that G has 2q edges but G contains no induced subgraphs of size g.

(sol)

Let $G = K_{m,n} \cup K_2$, where m and n are both odd,

$\| G \| = mn + 1 = 2q$.

由於兩個變數(m,n)不容易處理，所以考慮固定其中一個 partite set 的元素個數.

Let m = 5 (Since m = 1,3 can not do it.) and

let $G = K_{5,2k+1} \cup K_2$, $k \in \mathbb{N} \cup \{0\}$

$\Rightarrow \| G \| = 5(2k+1)+1 = 10k+6 = 2(5k+3) = 2q$.

(q = 5k+3, q-1 = 5k+2)

If G has an induced subgraph G* = (A,B) of size g, then one of the following condition is true. (Not both)

(1) For all $e \in E(G^*)$, $e \in E(K_{5,2k+1})$.

(2) There exists an egde $e \in E(G^*)$ such that $e \notin E(K_{5,2k+1})$, in fact, $e \in E(K_2)$.

In (1), G* has 5k+3 edges in $K_{5,2k+1}$.

In (2), G* has 5k+2 edges in $K_{5,2k+1}$.

Since $K_{5,2k+1}$ is a complete bipartite graph,

the vertices of $A\setminus\{x\}$, $x \in V(K_2)$, have the

same degree.

$\Rightarrow$ In (1), there exists $3 \le t \le 5$ such that $t \mid 5k+3$.

In (2), there exists $3 \le t \le 5$ such that $t \mid 5k+2$.

(t = 1, $1 \mid 5k+3$ and $1 \mid 5k+2$ are true,

but $2k+1 < 5k+3$ and $2k+1 < 5k+2$,

t = 2, $2 \mid 5k+3$ or $2 \mid 5k+2$ is true,

but $2k+1 < (5k+2)/2$

(and $2k+1 < (5k+3)/2$)

$\Rightarrow$ For t = 1 or 2, there does not exist induced

subgraph of size 5k+3 or 5k+2.)

By the above statement, we want the following condition

can be true.

" For $3 \le t \le 5$, t does not divide 5k+3 and

t does not divide 5k+2."

(1) t = 3 $\Rightarrow$ 3 does not divide 5k+3 and

3 does not divide 5k+2

$\Rightarrow k \neq 0, 2 \pmod 3$

(2) $t = 4 \Rightarrow 4$ does not divide $5k+3$ and

4 does not divide $5k+2$

$\Rightarrow k \neq 1, 2 \pmod 4$

(3) $t = 5 \Rightarrow 5$ does not divide $5k+3$ and

5 does not divide $5k+2$

$\Rightarrow$ It is true for all $k \in \mathbb{N} \cup \{0\}$.

By (1),(2),(3) $\Rightarrow k \equiv 4, 7 \pmod{12}$.

Hence, $\{G = K_{5,2k+1} \cup K_2 \mid k \equiv 4, 7 \pmod{12}\}$ are the desired graphs. $\square$

Graph Theory I　Homework I-2　by　李光祥

2. Prove that if a bipartite graph G has 16 edges, then
G contains an induced subgraph of size 8.

(pf)

Let G = (A,B) with $\|G\|$ = 16, where A = $\{v_1,v_2,\ldots,v_n\}$
and $deg_G(v_1) \geq deg_G(v_2) \geq \cdots\cdots \geq deg_G(v_n) \geq 1$, $1 \leq n \leq 16$.

Note：If G has a part in which there are some vertices
whose degree sum is 8, then these vertices induce
a subgraph of size 8.

We use a Lemma to prove this exercise.

Lemma (Chang and Juan, 2002)：

Suppose vertices x and y are in the same part of a
bipartite graph G.　If $deg_G(x) + deg_G(y) \geq 2m$, then
G has an induced subgraph with exactly 2m edges.

If $8 \leq deg_G(v_1) \leq 16$, then G has an induced subgraph
of size 8.($K_{1,8}$)

(1) $deg_G(v_1) = 7$

Then $deg_G(v_2) \in \{1,2,\ldots,7\} \Rightarrow deg_G(v_1) + deg_G(v_2) \geq 8$.

By Lemma, G has an induced subgraph of size 8.

(2) $\deg_G(v_1) = 6$

Then $\deg_G(v_2) \in \{1,2,\ldots,6\}$.

For $2 \le \deg_G(v_2) \le 6$, $\deg_G(v_1) + \deg_G(v_2) \ge 8$.

By Lemma, G has an induced subgraph of size 8.

When $\deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{6},\boxed{1},\boxed{1},\ldots,1)$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

(3) $\deg_G(v_1) = 5$

Then $\deg_G(v_2) \in \{1,2,\ldots,5\}$.

For $3 \le \deg_G(v_2) \le 5$, $\deg_G(v_1) + \deg_G(v_2) \ge 8$.

By Lemma, G has an induced subgraph of size 8.

When $\deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{5},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

When $\deg_G(v_2) = 2$, the degree sequence of part A

is $(\boxed{5},\boxed{2},\underline{\qquad\quad},\boxed{1})$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

(4) $\deg_G(v_1) = 4$

Then $\deg_G(v_2) \in \{1,2,3,4\}$.

For $\deg_G(v_2) = 4$, $\deg_G(v_1) + \deg_G(v_2) \geq 8$.

By Lemma, G has an induced subgraph of size 8.

When $\deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{4},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

When $\deg_G(v_2) = 2$, the degree sequence of part A

is $(\boxed{4},\boxed{2},\boxed{2},$______$)$ or $(\boxed{4},\boxed{2},\boxed{1},\boxed{1},$______$,1)$.

When $\deg_G(v_2) = 3$, the degree sequence of part A

is (4,3,3,3,3) (Special Case,最後再討論),

$(\boxed{4},\boxed{3},$______$,\boxed{1})$ or $(\boxed{4},3,3,2,\boxed{2},\boxed{2})$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

(5) $\deg_G(v_1) = 3$

Then $\deg_G(v_2) \in \{1,2,3\}$.

When $\deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

When $\deg_G(v_2) = 2$, the degree sequence of part A

is $(\boxed{3},\boxed{2},\boxed{2},$______$,\boxed{1})$ or $(\boxed{3},\boxed{2},\boxed{1},\boxed{1},\boxed{1},$______$,1)$.

When $\deg_G(v_2) = 3$, the degree sequence of part A is (3,3,3,3,3,1) (Special Case,最後再討論),

$(\boxed{3},\boxed{3},\_\_\_,\boxed{2},\_\_\_)$ or $(\boxed{3},\boxed{3},\_\_\_\_\_\_,\boxed{1},\boxed{1})$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

(6) $deg_G(v_1) = 2$

Then $deg_G(v_2) = 1$ or 2.

When $deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{2},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

When $deg_G(v_2) = 2$, the degree sequence of part A

is $(\boxed{2},\boxed{2},\boxed{2},\boxed{2},\_\_\_\_\_)$, $(\boxed{2},\boxed{2},\boxed{2},\boxed{1},\boxed{1},\ldots,1)$ or

$((\boxed{2},\boxed{2},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

(7) $deg_G(v_1) = 1$

Then $deg_G(v_2) = 1$, the degree sequence of part A

is $(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\ldots,1)$.

$\Rightarrow$ There are some vertices whose degree sum is 8.

$\Rightarrow$ G has an induced subgraph of size 8.

Now, we discuss two special cases (4,3,3,3,3) and

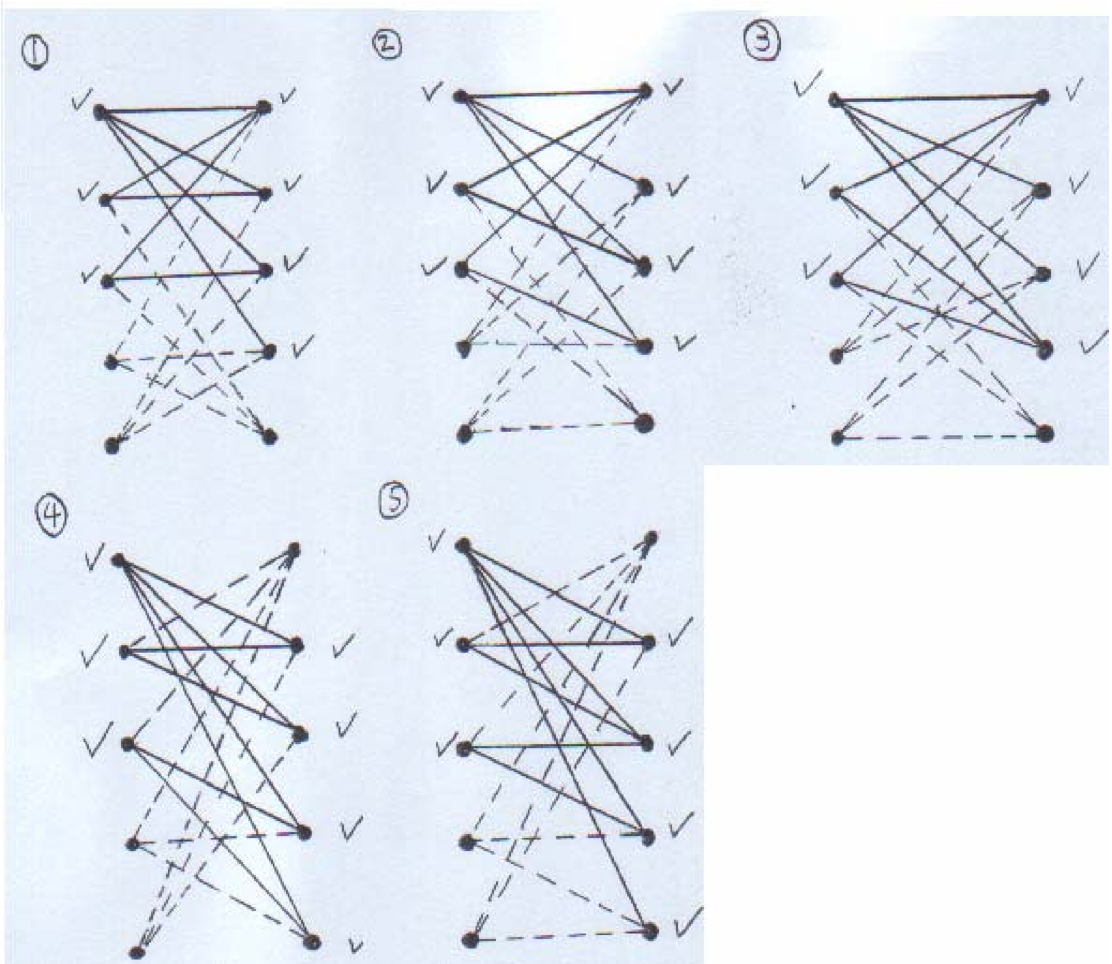
(3,3,3,3,3,1).

(1) Degree sequence of part A and part B are both

(4,3,3,3,3).

Then there are 5 cases (up to isomorphism) as

following (實線的邊：induced subgraph of size 8.)：

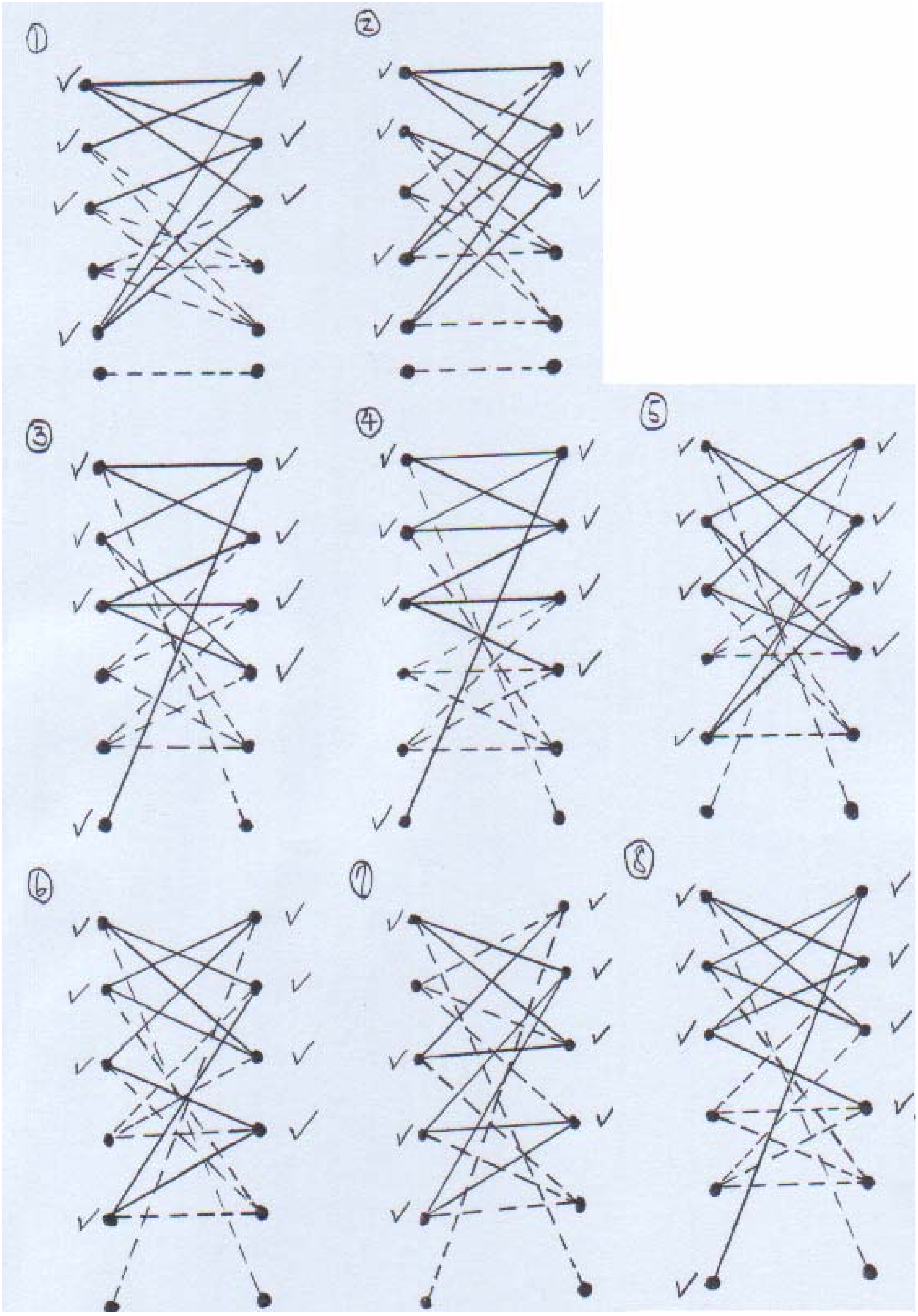


(2) Degree sequence of part A and part B are both

(3,3,3,3,3,1).

Then there are 8 cases (up to isomorphism) as

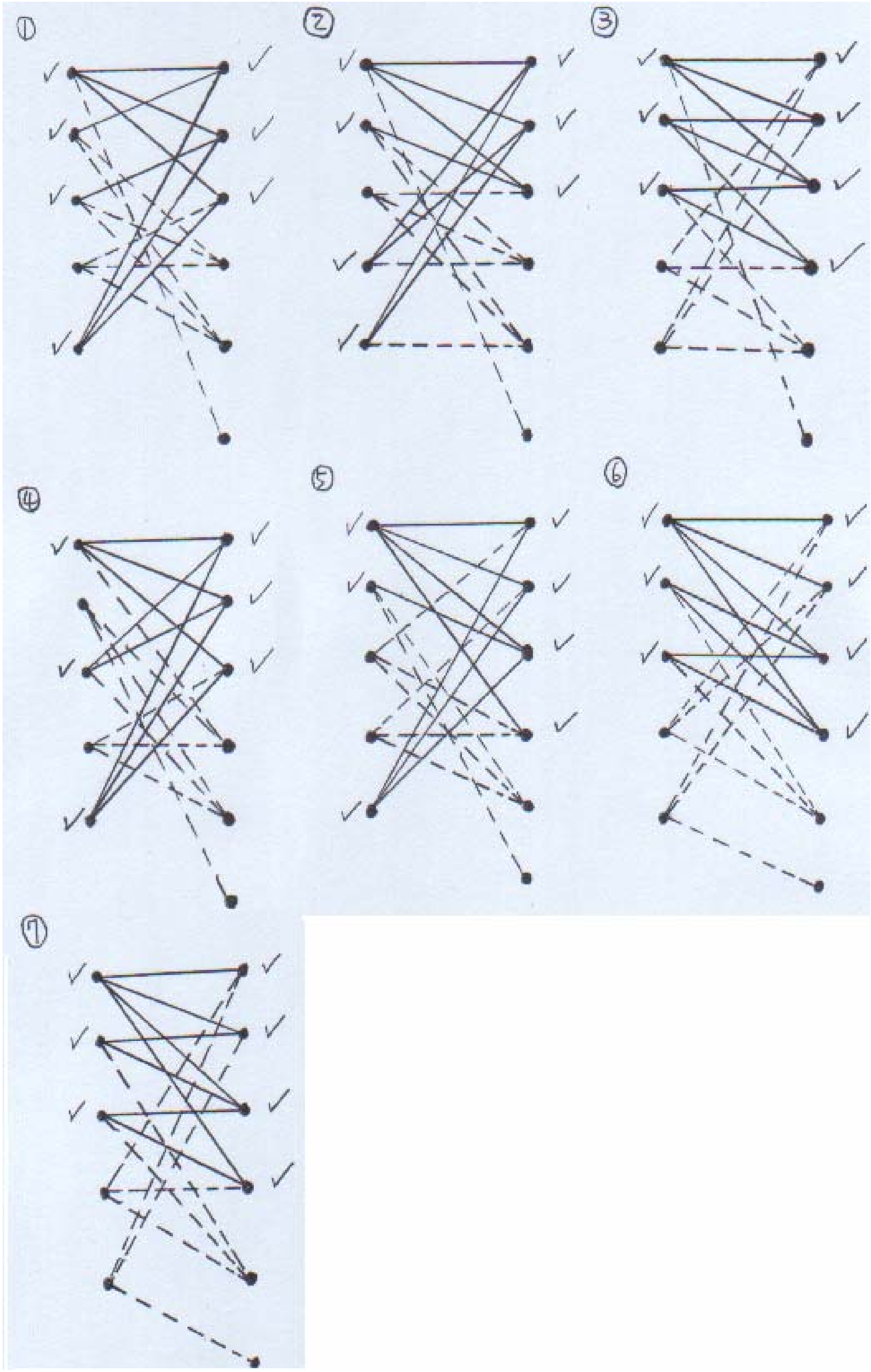
following (實線的邊：induced subgraph of size 8.)：



(2) WLOG, degree sequence of part A is (4,3,3,3,3),

degree sequence of part B is (3,3,3,3,3,1).

Then there are 7 cases (up to isomorphism) as following (實線的邊：induced subgraph of size 8.)：



Hence, this proof is done. □

Graph Theory I Homework I-3 by TA

Problem

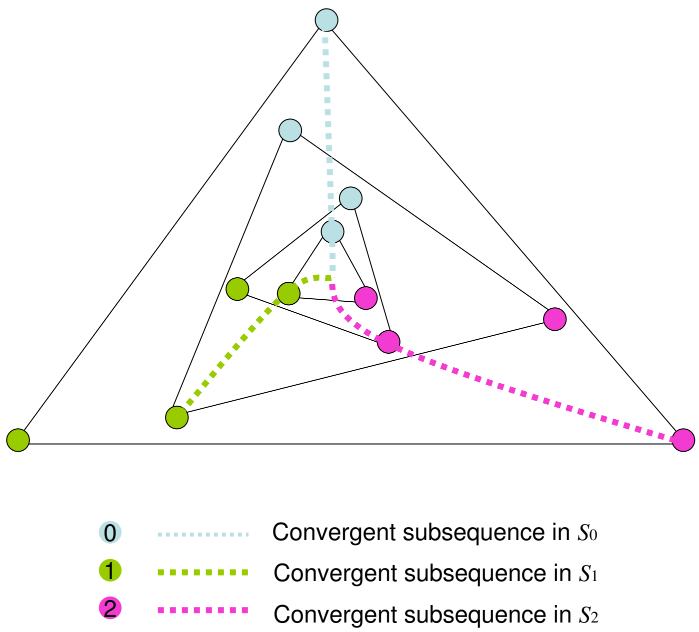
Use Sperner's Lemma to prove the Brouwer's Fixed Point Theorem for 2-dimension space.

Proof. Let points v_0, v_1, v_2 be the corners of T. We can express a point on a segment uniquely as a weighted average of its endpoints, so we can express each $v \in T$ uniquely as a weighted average of corners: $v = a_0v_0 + a_1v_1 + a_2v_2$, where $\sum a_i = 1$ and $a_i \geq 0$. Specify v by (a_0, a_1, a_2).

Let f be a continuous mapping from T to itself. Define $S_i = \{a; a'_i \leq a_i\}$ for $i = 1, 2, 3$, where $f(a) = a'$. Since the coefficients of each point sum to 1, every point in T belongs to some S_i, and a point belongs to all three sets if and only if it is a fixed point for f.

Claim: $\bigcap_{i=1,2,3} S_i \neq \emptyset$.

Proof of the claim: Given a simplicial subdivision of T, for each node a choose



a label i s.t. $a \in S_i$. Since points on the edge of T opposite v_i have ith coordinate 0, their ith coordinate connot decrease under f. This shows that their labels only belong to $\{1, 2, 3\} \setminus \{i\}$. Thus the labeling is proper, and Spener's Lemma guarantees a complete labeled cell. Repeating the process using triangulations with successively smaller cells yields a sequence of successively smaller completely labeled triangles. Let the jth triangle have corners x_j, y_j, z_j with labels $0, 1, 2$, respectively. In each S_i, we obtain a infinite sequence of points, say $(x_j), (y_j)$ and (z_j), respectively. Since f is continuous, each S_i is closed and bounded. Bolzano-Weierstrass theorem says that every infinite sequence of points in a closed and bounded set has a convergent subsequence convergent to a point in that set. Let (x_{i_k}) be a convergent subsequence of (x_i) and similarly for (y_{i_k}) and (z_{i_k}). Because the distance from x_{i_k} to y_{i_k} and (z_{i_k}) approaches to 0, they converge to the same point and this point belongs to S_i for $i = 1, 2, 3$ by BolzanoVWeierstrass theorem. We have the claim. ∎

By the claim we have there is a fixed point for f.

∎

Graph Theory I Homework I-4 by 游舜婷

4. *Use the adjacency matrix of a graph G to find the number of 5-cycles in G*

<Theorem> *The number of distinct walks with length t from v_i to v_j is equal to the (i, j) entry of A^t*

pf : Let $|G| = n$

By theorem we have the number of distinct walks with length 5 from v_i to v_j for all i is $\sum_{i=1}^{n} a^{(5)}_{ii} = tr(A^5)$

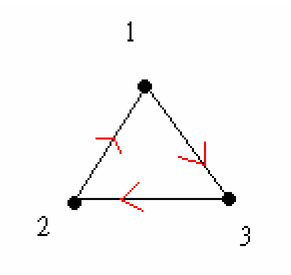
Now, we want to find the number of 5-cycles in G, we should removed two cases in the following:

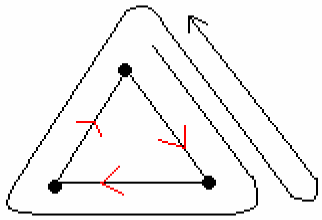
($\because$ doesn't exist closed 5-walks with 2 or 1 edges

$\therefore$ only consider these two cases)

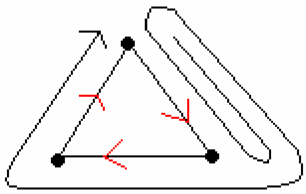
<Case1> colsed 5-walks with three edges

Given

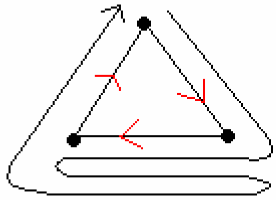




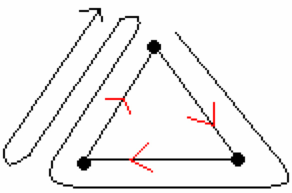
1-3-2-1-3-1



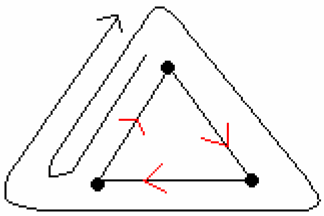
1-3-1-3-2-1



1-3-2-3-2-1



1-3-2-1-2-1



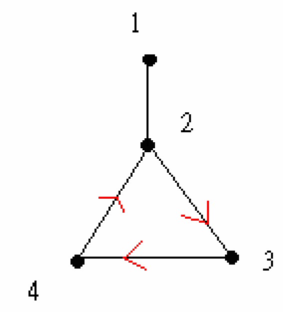
1-2-1-3-2-1

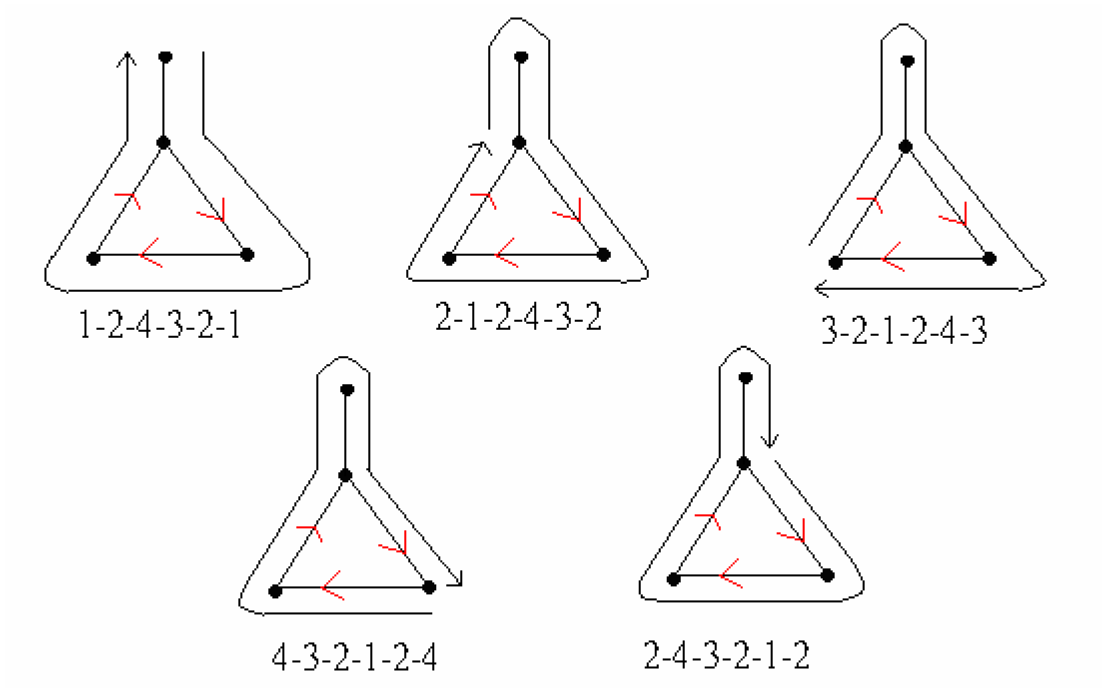
$\therefore$ *the number of distinct closed* 5-*walks with three edges*

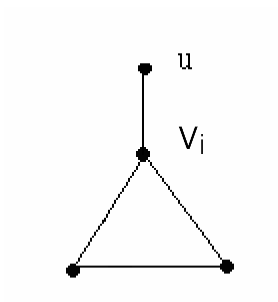
$$is \;\; 5\times\sum_{i=1}^{n} a_{ii}^{(3)} = 5tr(A^3)$$

<*Case*2> *closed* 5-*walks with four edges*

Given







How many vertex u can choose?

$\Rightarrow \deg(v_i) - 2 \quad (\text{"2"} \rightarrow \text{ two edges of the triangle})$

$$\Rightarrow \left(\sum_{j=1}^{n} a_{ij}\right) - 2$$

Hence the number of distinct closed 5-*walks with* 4 *edges is*

$$5\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} a_{ij}\right) - 2\right] a_{ii}^{(3)}$$

$\therefore$ *by case*1 *and case*2 *we have the number of* 5-*cycle is*

$$\frac{1}{10}\{tr(A^5)-5tr(A^3)-5\sum_{i=1}^{n}[(\sum_{j=1}^{n}a_{ij})-2]a_{ii}^{(3)}\}$$

(每一個*walk*可從一個點的左邊走，亦可從右邊走

$\because$ 5$-$*cycle* 有五個點　$\therefore$同一個*cycle*共被算了$2\times5=10$次)

Graph Theory I　Homework I-5　by　黃信菖

Problem:

Let G be a graph with 100 vertices. Find a regular graph of diameter 6 with as less edges as possible.

Proof.

令$G=(V,E)$，其中$|V|=100$。將所有 node 編號 0 至 99 號，其 node 與 node 之間的連法爲：

(1) If $i=4k$ or $4k-1$，則$i(i\pm 1 \bmod 100)$與$i(i\pm 8 \bmod 100)\in E$

(2) If $i=4k+1$ or $4k+2$，則$i(i\pm 1 \bmod 100)$與$i(i\pm 16 \bmod 100)\in E$

($k\in Z^{+}\cup\{0\}$)

所以 0 會與 1、8、99、92 相連，1 會與 0、2、17、85 相連…

$\Rightarrow G$ 爲 4-regular

每個 node 除了會與相鄰編號的 node 相連之外，還會有 4 個C_{25}：

1. $0-8-16-24-32-40-48-56-64-72-80-88-96-4-12-20-28$
$-36-44-52-60-68-76-84-92-0$
2. $1-17-33-49-65-81-97-13-29-45-61-77-93-9-25-41-57$
$-73-89-5-21-37-53-69-85-1$
3. $2-18-34-50-66-82-98-14-30-46-62-78-94-10-26-42-58$
$-74-90-6-22-38-54-70-86-2$
4. $3-11-19-27-35-43-51-59-67-75-83-91-99-7-15-23-31$
$-39-47-55-63-71-79-87-95-3$

共 200 個邊

確認任兩個 node 在 6 步內可相連，且最遠距離爲 6。

首先，node 0 與 node $4k$、$4k-1$點對稱，且 node 0 走到i與$100-i$爲相反路徑、故只需 check node 0 可以走到 node 1 ~ 50 即可。

0 至 50：$0-1-17-33-49-50$

∟ $48-47$

0 至 46：$0-99-98-14-30-46$

∟45

0 至 44：$0-1-85-59-53-52-44$　(44 爲最遠距離)

0 至 43：$0-8-9-25-41-42-43$

0 至 40：$0-8-16-24-32-40$

∟39

0 至 38：$0-1-85-69-53-37-38$

0 至 36：$0-1-17-33-34-35-36$

0 至 32：0 – 8 – 16 – 24 – 32
∟ 31 – 30
0 至 29：0 – 99 – 98 – 14 – 30 – 29
∟ 28
0 至 27：0 – 8 – 16 – 24 – 25 – 26 – 27
∟ 23 – 22 – 21
0 至 20：0 – 8 – 16 – 17 – 18 – 19 – 20
∟ 15 – 14 – 13 – 12
0 至 11：0 – 8 – 9 – 10 – 11
∟ 7 – 6 – 5 – 4
0 至 3：0 – 1 – 2 – 3

同理，node 1 對稱於 node 2，且走到 i 與 $101 - i$ 爲相反路徑、故只需 check node 1 可以走到 node 2~ 51 即可。

1 至 51：1 – 17 – 33 – 49 – 50 – 51
∟ 48 – 47 – 46
1 至 45：1 – 85 – 69 – 53 – 52 – 44 – 45 （45 爲最遠距離）
∟ 43
1 至 42：1 – 17 – 33 – 32 – 40 – 41 – 42
∟ 39 – 38
1 至 37：1 – 17 – 33 – 34 – 35 – 36 – 37
∟ 32 – 31 – 30 – 29
1 至 28：1 – 17 – 18 – 19 – 20 – 28
∟ 27
1 至 26：1 – 17 – 16 – 24 – 25 – 26
∟ 23 – 22 – 21
1 至 16：1 – 17 – 16 – 15 – 14 – 13 – 12
1 至 11：1 – 0 – 8 – 9 – 10 – 11
∟ 7 – 6 – 5 – 4
1 至 3：1 – 2 – 3

Graph Theory I Homework I-8 by 李柏瑩

Problem:

The Odd Graph O_k . Prove that the girth of O_k is 6 if $k \geq 3$.

Proof:

Step1: Claim: the girth of O_k $\geq$ 6

(i) O_k has no 1-cycle or 2-cycle(Since O_k is a simple graph)

(ii) O_k has no 3-cycle

[If O_k has a 3-cycle $\Rightarrow$ there exists three distinct k-element subsets of {1,2,…,2k+1}such that they are pairwise disjoint. But it is impossible!

(Since W.L.O.G..我們可以選前兩個頂點分別爲{ 1, 2,…, k},{k+1,k+2,…,2k}則第三個頂點只剩下 2k+1 這個元素可選)]

(iii) O_k has no 4-cycle

[If O_k has a 4-cycle. W.L.O.G.. we can choose V_1={ 1, 2,…, k} ,V_2 ={k+1,k+2,…,2k}in figure 1

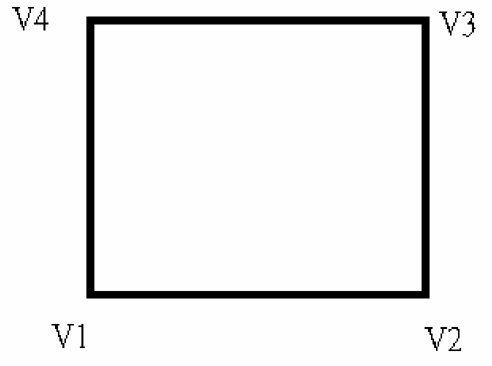


Figure 1

then we may choose V_3={1,2,.,k-1,2k+1} $\neq V_1$

$\Rightarrow V_4$中的元素只能從 k+1 到 2k 中挑,但V_4又不能等於

V_2={k+1,k+2,…,2k}

$\Rightarrow V_4$不存在→←

(iv) O_k has no 5-cycle. W.L.O.G.. we can choose V_1={ 1, 2,…,

k} ,V_2 ={k+1,k+2,…,2k},V_3={1,2,.,k-1,2k+1},V_4={k,k+1,…,2k-1}in figure 2

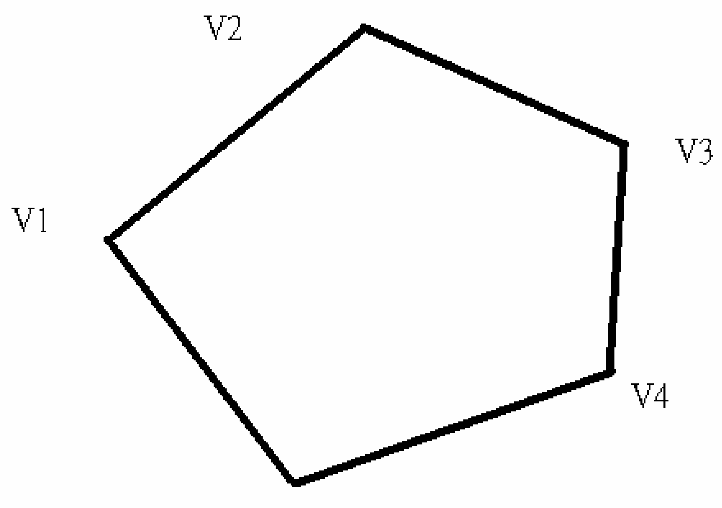


Figure 2

$\Rightarrow V_5$中的元素只有 2k 和 2k+1 可挑

$\Rightarrow V_5$不存在→←

Step2: Claim: the girth of O_k $\leq$ 6

We will construct a 6-cycle in O_k

Let V_1={ 1, 2,…, k},V_2 = {k+1,k+2,…,2k},V_3={1,2,.,k-1,2k+1}

V_4= {k, k+1,…,2k-1},V_5={1,2,.,k-1,2k}, V_6={k+1,k+2,…,2k-1,2k+1}in figure 3

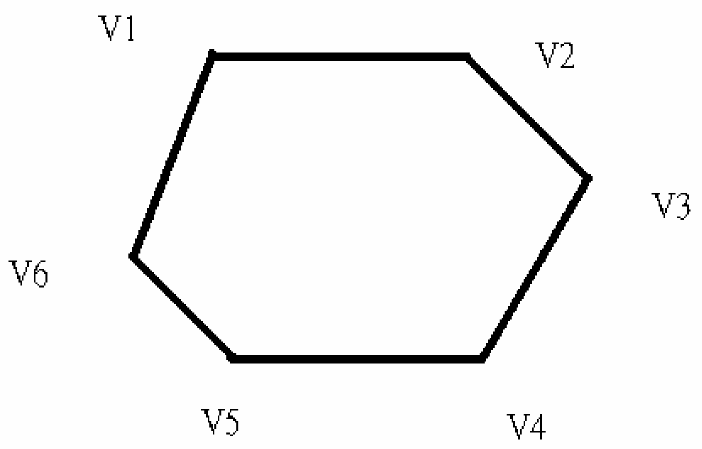


Figure 3

$\Rightarrow$ We can construct a 6-cycle in O_k

By Step1& Step 2 $\Rightarrow$ the girth of O_k =6, $\forall$ k $\geq$ 3

Graph Theory I　Homework I-7　by　黃信菖

Problem:

Prove that a self-complementary graph with *n* vertices exists if and only if *n* or *n*-1 is divided by 4.

Proof:

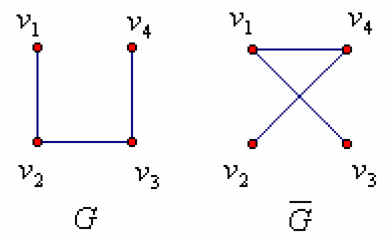
($\Rightarrow$) $\|\overline{G}\| = \|G\|$ and $\|\overline{G}\| + \|G\| = \binom{n}{2}$

$\Rightarrow \|G\| = \dfrac{n(n-1)}{4}$

$\therefore n = 0 \ or \ 1 \pmod 4$

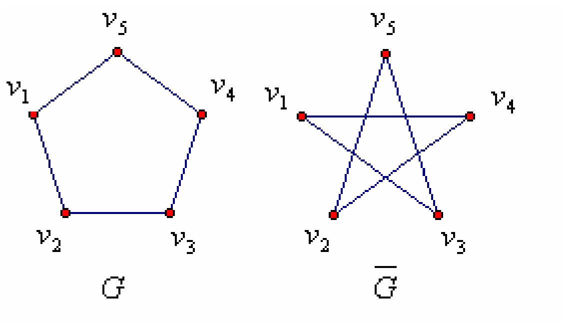
($\Leftarrow$) By induction on $n = 4k \ and \ 4k+1$

$k = 1 \Rightarrow n = 4$ ：



It is easy to see G isomorphic to $\overline{G}$

$n = 5$ ：



It is easy to see G isomorphic to $\overline{G}$

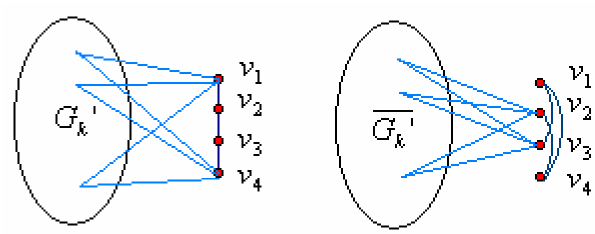
Assume it hold on $n = 4k' \ and \ 4k'+1$

then when $k = k'+1$

$\Rightarrow n = 4k'+4$ ： because $K_{4k'+4} = K_{4k'} \vee K_4$

by induction 可知，$K_{4k'}$ 存在 self-complement graph

G_k' 與 $\overline{G_k'}$，所以將 $K_{4k'} \vee K_4$ 拆成兩個部份，如下：



其中將 K_4 拆爲 2 條 P_4，其頭尾各連接 G_k' 與 $\overline{G_k'}$ 的

各個點。故此兩個 part 互爲 isomorphic

$$\text{其邊數和} = 2\left(\frac{\binom{4k'}{2}}{2} + 3 + 4k' \times 2\right)$$

$$= \frac{(4k')^2 - 4k' + 12 + 32k'}{2}$$

$$= \binom{4k'+4}{2} = \| K_{4k'+4} \|$$

故 $K_{4k'+4}$ 存在 self-complement graph

同理，當 $n = 4k'+5$ 時，$K_{4k'+5} = K_{4k'+1} \vee K_4$ 仿造上述的方法，故 $K_{4k'+5}$ 亦存在 self-complement graph。

Graph Theory I Homework I-8 by 劉宜君

8. Prove that if $\frac{p(p-1)}{2}-x$ is a multiple of 3 and $x\in\{0,1,2\}$, then $K_p - P_{x+1}$ can be decomposed into three isomorphic subgraphs. ($p\geq 3$)

Pf.

Suppose $G = G_1 \cup G_2 \cup G_3$ and $G_1 \cong G_2 \cong G_3$, and we know that K_{2n} can be decomposed into n Hamiltonian paths. Let $K_6 = H_{14} \cup H_{25} \cup H_{36}$, where H_{ij} is a Hamiltonian path that start from v_i and end at v_j. ($H_{14} \cong H_{25} \cong H_{36}$)

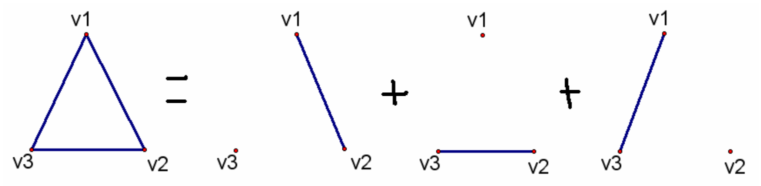
We define the $|G|+6$ construction is $G \vee K_6 = G_1^{'} \cup G_2^{'} \cup G_3^{'}$ where

$G_i^{'} = \{G_i \cup H_{i,i+3} \cup \{v_i \sim v \quad and \quad v_{i+3} \sim v \mid \forall v \in G\}\}$, i=1,2,3. Clearly, $G_1^{'} \cong G_2^{'} \cong G_3^{'}$.

⇨ If we can prove that graph G with order p can be decomposed into three isomorphic subgroups, then we also prove that $G \vee K_6$ can be decomposed into three isomorphic subgroups. So we have to consider those graph with size p=3～8. And since $\forall p \in \mathrm{N}$, $p=3k$、$p=3k+1$ or $p=3k+2$, $\forall k \in \mathrm{N} \cup \{0\}$, the $3 \left| \frac{p(p-1)}{2} - x \right.$, x will be 0 or 1.

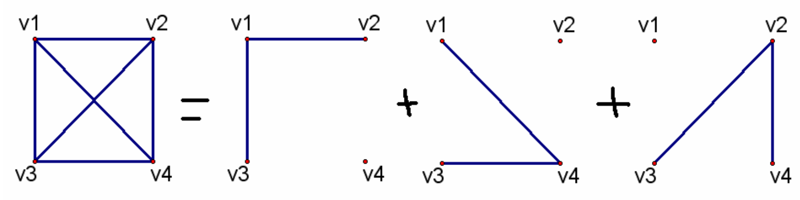
Consider x=0,

P=3, K_3



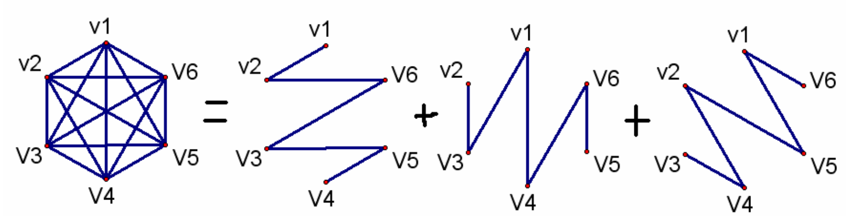
=>P=3+6k, $\forall k \in \mathrm{N} \cup \{0\}$ done.

P=4, K_4:



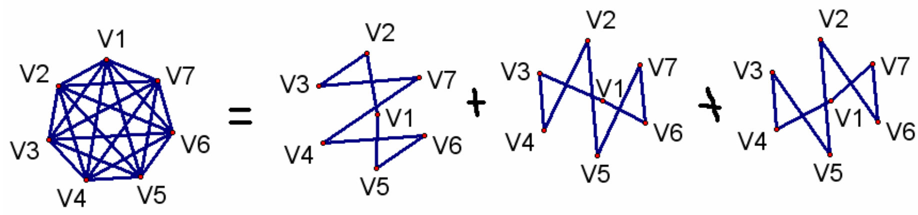
=>P=4+6k, $\forall k \in \mathrm{N} \cup \{0\}$ done.

P=6,
K_6



=>P=6+6k, $\forall k \in \mathrm{N} \cup \{0\}$ done.

P=7,
K_7



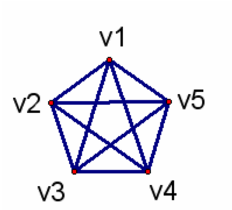
=>P=7+6k, $\forall k \in \mathrm{N} \cup \{0\}$ done.

Consider x=1, if $K_p - P_2$ can be decomposed into three isomorphic subgroups,

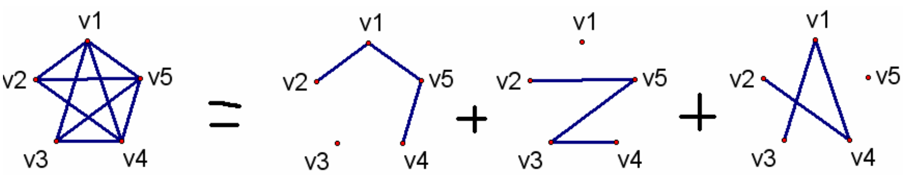
then if let $K_{p+6} - P_2 = (K_p - P_2) \vee K_6$ (delete the same edge with K_p),

then by $|G| + 6$ construction $K_{p+6} - P_2$ can be decomposed into three isomorphic subgroups.

P=5,
K_5

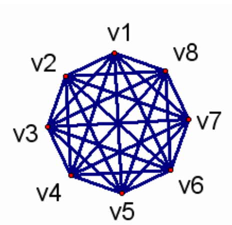


$K_5 \text{-} e_{v_2 v_3}$

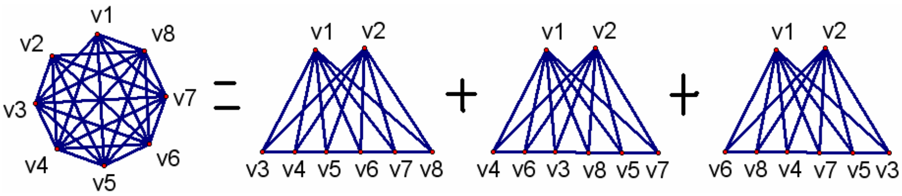


=>P=5+6k, $\forall k \in \mathrm{N} \cup \{0\}$ done.

P=8,
K_8



$K_8 - e_{v_1 v_2}$

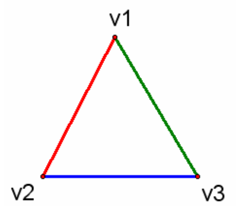


$\Rightarrow P=8+6k, \quad \forall k \in \mathrm{N} \cup \{0\}$ done.

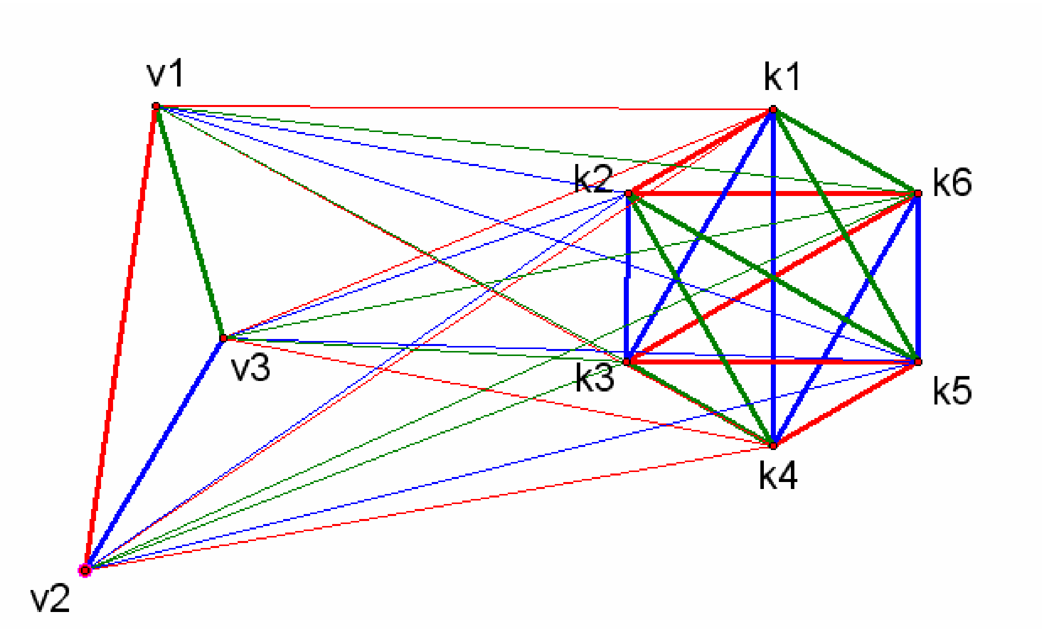
So we prove that $\forall p \in \mathrm{N} \setminus \{1,2\}$, if $3 \left| \frac{p(p-1)}{2} - x \right.$, $x \in \{0,1\}$, then $K_p - P_{x+1}$ can be decomposed into three isomorphic subgraphs.

An example of $|G|+6$ construction：

K_3



$|G|+6$ construction



This graph is K_{3+6} and can be decomposed into three isomorphic subgraphs.
(different colors)

Graph Theory I　Homework I-9　by　曾慧棻

9. For k ≥ 2 and g ≥ 2, prove that there exists a k-regular graph with girth g.

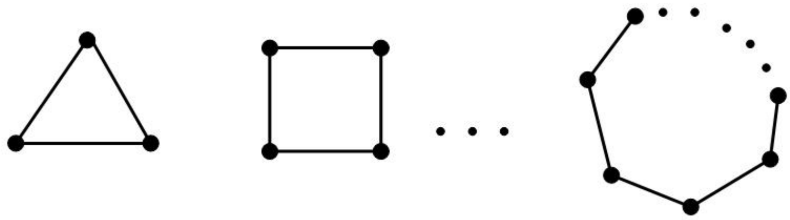
Pf: By induction on k and g

It's easy to know, when k=2, g=2 is true.

First , induction on g.

We know (2,g)-graph exists for all g

Since



(2,3)-regular　　(2,4)-regular　　(2,g)-regular
=a cycle of length g

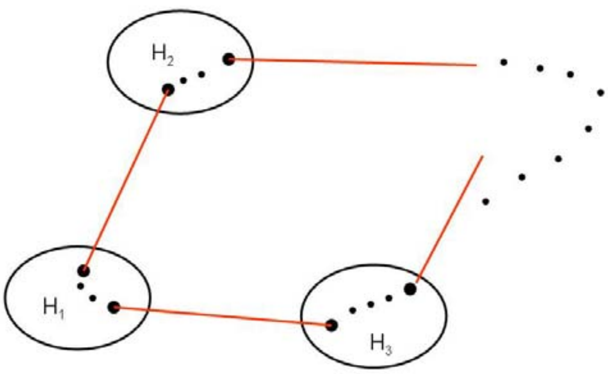
Second, induction on k

Suppose the assertion is true for all pairs (k',g')-graph where k'< k and g'< g

Let H be a (k-1,g)-graph and G' be a (|V(H)| , $\left\lceil \frac{g}{2} \right\rceil$)-graph, we use H and G' to construct G to be a (k,g)-graph. We construct a G'. Since G' has |V(H)|-vertex, 我們把它每一個點都取代成一個H and since G' is |V(H)|-regular graph. Hence H上的每一個vertex都可以多一個edge連出去 such that deg(v_i)=k , for all i in G.

Next, since G' has girth =$\left\lceil \frac{g}{2} \right\rceil$, we find a cycle with girth=$\left\lceil \frac{g}{2} \right\rceil$ and add girth=g of H.

See Fig.



Suppose we find a cycle of girth=$\left\lceil \frac{g}{2} \right\rceil$, since deg=|V(H)| in G', 因此在H的點都可以對外連出, 且每個點連出去的H_i都是不一樣. 假如在H_i 中連出去兩點,最短path is only an edge.那原本 cycle的girth=$\left\lceil \frac{g}{2} \right\rceil$ 再加上在H_i 內的連接邊數

For G , girth ≥ $\left\lceil \frac{g}{2} \right\rceil$*2=g , if the girth > g in this cycle, we take girth=g in H.

Hence, there exists k-regular graph with girth g . We have done.

Graph Theory I Homework I-10 by 劉士慶

10、(a) **Use Proposition 1.3.11 to prove that if G has more than n^2/4 edges, then G has a vertex whose deletion leaves a graph with more than (n-1)^2/4 edges.**

(b) Use part(a) to prove by induction that G contains a triangle if e(G)> n^2/4.

Sol. (a) Since e(G) is an integer, we have

$$\begin{cases} e(G) \geqq (n^2+4)/4, & n \text{ is even.} \\ e(G) \geqq (n^2+3)/4, & n \text{ is odd.} \end{cases}$$

⟹ we have e(G)≧ (n^2+3)/4.

By proposition 1.3.11, we know that e(G)= $\sum_{i=1\sim n}$ e(G-v_i)/(n-2).

⟹ e(G)= $\sum_{i=1\sim n}$ e(G-v_i)/(n-2)≧ (n^2+3)/4.

⟹ 左式裡最大的 term 至少要大於右式的平均值.

⟹ there is a vertex v such that e(G-v)/(n-2)≧ 1/n*(n^2+3)/4.

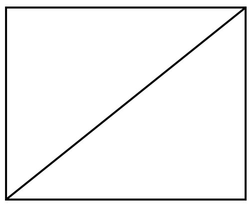
⟹ e(G-v)≧ (n-2)*(n^2+3)/4n=(n^3-2n^2+3n-6)/4n

=(n^2-2n+1)/4+(2n-6)/4.

Since n≧4 ⟹ (2n-6)/4> 0.

Hence we get the inequality e(G-v)> (n-1)^2/4.

(b)Use induction on n.

k=4,  ,由圖可得知,e(G)≧5 必有 △ .

∴e(G)> 4.

Then assuming that k= n, G contains a triangle when e(G)>n^2/4.

When k= n+1, e(G)>(n+1)^2/4, by (a), there exists a vertex v such that e(G\{v})> n^2/4 $\Longrightarrow$ by induction hypothesis G-{v} contains a triangle.

Hence G contains a triangle.