Graph Theory(I) - Final Exam

1. Prove that for each $g \geq 3$, a $(3,g)$-cage has no bridge.

Pf.

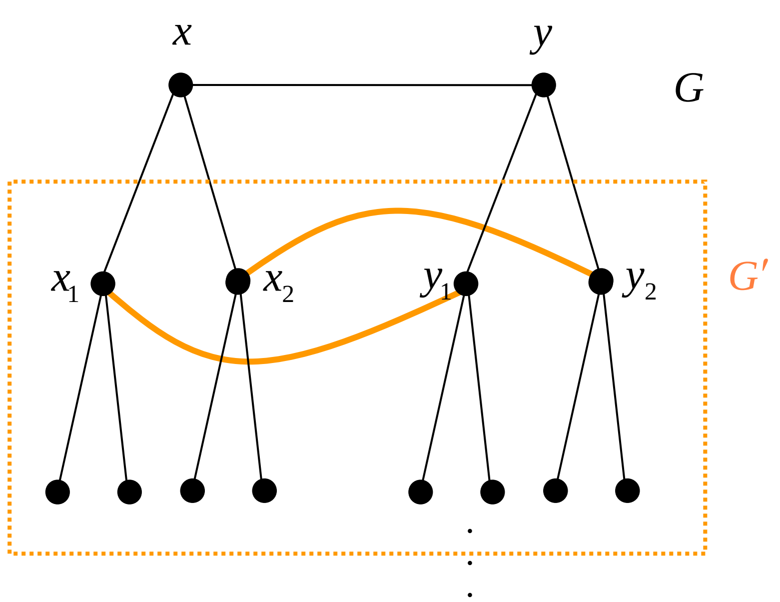
Assume we have known the lemma below: (skipping the proof)

Lemma $\forall k \geq 3$ and $3 \leq g_1 < g_2 \Rightarrow f(k;g_1) < f(k;g_2)$, where

$f(k;g)$ is the number of vertices of a (k,g)-cage.

Suppose there is a bridge $e(x,y)$ in a $(3,g)$-cage G. Then $G-e(x,y)$ has 2 components, called G_1 and G_2. W.L.O.G., let $x \in V(G_1)$ and $y \in V(G_2)$. Then $G_1 - x$ has two vertices x_1, x_2 with degree 2, and $G_2 - y$ also has y_1, y_2 with degree 2. Then we add $e(x_1,y_1)$ and $e(x_2,y_2)$ to connect $G_1 - x$ and $G_2 - y$, and obtain a new graph G'. Then G' is 3-regular and $g(G') \geq g(G)$, but $|V(G')| = |V(G)| - 2 < |V(G)|$, a contradiction.

Hence, a $(3,g)$-cage has no bridge! ■



Graph Theory(I) - Final Exam

2. Let $k \geq 2$. Show that every k-linked graph of order at least $4k$ contains a cycle of length at least $4k-2$.

Proof.

Let G be a k-linked graph of order at least $4k$. Clearly, G is $2k-1$ connected.

Suppose $C = \{v_1, v_2, \cdots, v_l\}$ is the longest cycle in G with length l less than $4k-2$. Let v be a vertex in $V(G) \setminus C$. Then, there exist at least either $2k-1$ or as many as $|C| = l$ internally disjoint paths from v to C since G is $2k-1$ connected. By the pigeonhole principle, there are two endpoints v_i, v_j of these paths such that v_i and v_j are adjacent in C for some i, j. Therefore, we can extend the original cycle C to a longer cycle $C' = \{v_1, \cdots, v_i, v, v_j, \cdots, v_l\}$ by adding the vertex v, a contradiction to the assumption that C is the longest cycle.

Graph Theory(I) - Final Exam

3. For each graph G of order p, prove that $\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = p$.

$\alpha(G)$：minimum size of vertex cover
$\beta(G)$：maximum size of independent set
$\alpha_1(G)$：minimum size of edge cover
$\beta_1(G)$：maximum size of matching

[proof]

1° claim $\alpha(G) + \beta(G) = p$.

Let A be a maximal independent set of V(G) $\Rightarrow$ $|A| = \beta(G)$
Let $B = V(G) - A$.
$\forall uv \in E(G)$, u 與 v 至少有一點在 B 中
$\Rightarrow$ B 爲 G 的一個 vertex cover. (因爲所有的邊都至少有一點在 B 裡面)
$\Rightarrow |B| \geq \alpha(G)$
$\therefore p = |A| + |B| \geq \alpha(G) + \beta(G)$ …… (♥)

Let C be a minimum vertex cover $\Rightarrow$ $|C| = \alpha(G)$
Let $D = V(G) - C$
$\Rightarrow$ D 中沒有邊了!! (因爲 D 中的每個點都不相鄰)
$\Rightarrow$ D 爲 G 的一個 independent set.
$\Rightarrow |D| \leq \beta(G)$
$\therefore p = |C| + |D| \leq \alpha(G) + \beta(G)$ …… (♥♥)

由(♥)與(♥♥)可知, $\alpha(G) + \beta(G) = p$.

2° claim $\alpha_1(G) + \beta_1(G) = p$.

Let A be a maximum matching $\Rightarrow$ $|A| = \beta_1(G)$
Let $B = A + \{ uv \mid u \in V(G) - V(A), v \in N_G(u)\}$
$\Rightarrow$ B 爲 G 中的一個 edge cover. (因爲 B 把 G 中所有的點都 cover 住了)
$\Rightarrow |B| \geq \alpha_1(G)$

又 A 為 matching, B 為 matching + 沒用到的點
故 $p = |A| + |B|$ (因為 matching 中一個邊用到兩個點)
$\therefore\ p = |A| + |B| \geq \alpha_1(G) + \beta_1(G)$ …… (♣)

Let C be a minimum edge cover $\Rightarrow$ $|C| = \alpha_1(G)$
$\Rightarrow$ C 中的每個 component 都是 star
設 C 有 k 個 components
在每個 star 中取一個邊, 就可造出一個 matching D
$\Rightarrow |D| \leq \beta_1(G)$, and $|D| = k$.
又 $p = k + |C|$ (k 表示 star 的中心點, $|C|$ 代表 star 除了中心點外的點數)
$\therefore\ p = |C| + |D| \leq \alpha_1(G) + \beta_1(G)$ …… (♣♣)

由(♣)與(♣♣)可知, $\alpha_1(G) + \beta_1(G) = p$.

3° 由 1°與 2°知, $\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = p$. (◕‿◕)

Graph Theory(I) - Final Exam

4. Prove that the minimum degree of G is k

if G is minimally k-edge-connected.

Proof: Consider a minimal set **S** s.t. $\left\|[S,\overline{S}]\right\| = k$.

Case 1: |S| =1

=> Let S={v} , then d(v) = k

又 $\delta(G) \geq \kappa'(G) = k$

=> $\delta(G) = k$ □

Case 2: $|S| \neq 1$

1) claim: G[S] connected

Pf: If G[S] is not connected

Let $S_1 \subset S$, G[S_1] is a component of G[S]

Then $\rightarrow\leftarrow$ to S : minimal set. □

2) $\therefore E(G[S]) \neq \varnothing$

Take $G' = G - \{e\}$, $e = xy \in E(G[S])$

=> G': k' edge connected , $k' < k$

=> $\exists T$, $\left\|[T,\overline{T}]\right\| = k' \leq k-1$ in G'.

3) claim:

(a) $\left\|[T,\overline{T}]\right\| = k-1$ in G'

(b) $\left\|[T,\overline{T}]\right\| = k$ in G

(c) $S \cap T \neq \varnothing$

(d) $S \cup T \neq V(G)$

Pf:

(a) if $\left\|\left[T,\overline{T}\right]\right\| < k-1$ in $G' = G - \{e\}$

Then $\left\|\left[T,\overline{T}\right]\right\| < k$ in G, a contradiction to G is k edge connected .

(b) if $\{x,y\} \in T$ (similar for $\{x,y\} \notin T \Rightarrow \{x,y\} \notin \overline{T}$)

Then T is also a set s.t. $\left\|\left[T,\overline{T}\right]\right\| = k-1$ in G , $\not\rightarrow$

$\therefore$ w.l.o.g. Let $\{x\} \in T$ and $\{y\} \notin T$.

$\because \left\|\left[T,\overline{T}\right]\right\| = k-1$ in G'

$\therefore \left\|\left[T,\overline{T}\right]\right\| = k$ in G □

(c) by (b)

$\therefore S \cap T \supseteq \{x\} \neq \varnothing$ □

(d) if $S \cup T = V(G)$

Then $T = V(G) - S'$, $S' \subset S$

$\Rightarrow \overline{T} = S'$, $|\overline{T}| = |S'| < |S|$

又 $\left\|\left[\overline{T},T\right]\right\| = \left\|\left[T,\overline{T}\right]\right\| = k$ in G

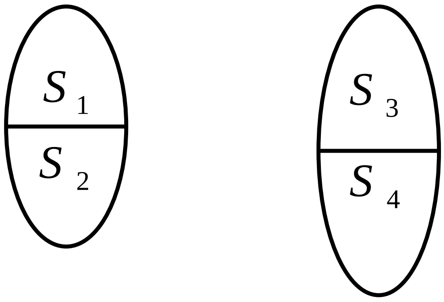
$\rightarrow\leftarrow$ to S : minimal. □

4) claim: For $S \subseteq V(G)$, let $d(s) = \left\|\left[S,\overline{S}\right]\right\|$

Let X,Y be nonempty proper vertex subsets of G.

Then $d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y)$

Pf: Let G:



$$S_1 \cup S_2 \cup S_3 \cup S_4 = V(G)$$

$$S_1 \cup S_2 = X \qquad S_1 \cup S_3 = Y$$

$$S_1 = X \cap Y \quad S_1 \cup S_2 \cup S_3 = X \cup Y$$

=> $d(X \cap Y) = \|[S_1, S_2]\| + \|[S_1, S_3]\| + \|[S_1, S_4]\|$

$$d(X \cup Y) = \|[S_4, S_1]\| + \|[S_4, S_2]\| + \|[S_4, S_3]\|$$

$$d(X) = \|[S_1, S_3]\| + \|[S_1, S_4]\| + \|[S_2, S_3]\| + \|[S_2, S_4]\|$$

$$d(Y) = \|[S_1, S_2]\| + \|[S_1, S_4]\| + \|[S_2, S_3]\| + \|[S_3, S_4]\|$$

=> $d(X) + d(Y) - d(X \cap Y) - d(X \cup Y) = 2\|[S_2, S_3]\| \geq 0$

$\square$

5) By 前

$$2k < d(S \cap T) + d(S \cup T)$$
$$\leq d(S) + d(T) = 2k \qquad \rightarrow\leftarrow$$

$\therefore$ Q.E.D. $\square$

ps: $\because \varnothing \neq (S \cap T) \subset S \quad \therefore d(S \cap T) > k$

$\because (S \cup T) \neq V(G) \quad \therefore d(S \cup T) \geq k$

Graph Theory(I) - Final Exam

5. Let D be a directed graph such that for each $v \in V(D)$ **,**

$\deg_D^+(v) = \deg_D^-(v) = t \geq 1$ **.**

Prove that D can be decomposed into t directed 1-factors.(with indegree 1 and outdegree1)

Proof:

Assume D is connected and $V(D) = \{v_1, v_2, v_3, ...v_n\}$

Otherwise, consider each component of D

Since $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v) = t \geq 1$

$\Rightarrow$ D is strong connected.

$\Rightarrow$ D has a directed E.C

Called C: $x_1 \to x_2 \to x_3 \to ..... \to x_q$ where $q = \|D\|$

Define a bipartite graph H with two partite set A,B

Where $A = \{a_1, a_2, a_3, ...a_n\}$, $B = \{b_1, b_2, b_3, ...b_n\}$

$a_{x_i} \leftrightarrow b_{x_j}$ in H $\Leftrightarrow$ $x_i \to x_j$ in C

$\Rightarrow$ H is a t-regular bipartite graph

By König's Theorem

$\Rightarrow$ H has an 1-factor M

$\Rightarrow$ M in H corresponds to a directed 1-factor in D

$\Rightarrow$ D can be decomposed into t 1-factors.

Graph Theory(I) - Final Exam

6. Prove that if $\|G\| = \binom{n+1}{2}$ and $\Delta(G) \leq \lfloor n/2 \rfloor$, then G can be decomposed into n subgraphs $G_1, G_2, \cdots, G_n$ such that G_i is induced by a matching with i edges for all i.

Lemma 1: Every k-edge colorable graph has an equitable k-edge coloring.

Proof.

Case 1: n is even.

By Vizing's Theorem and Lemma 1, G has an equitable $(\frac{n}{2}+1)$-edge coloring with color classes $A_1, A_2, \cdots, A_{\frac{n}{2}+1}$ s.t. $|A_i| = n-1$ for $i = 1, \cdots, \frac{n}{2}$ and $|A_{\frac{n}{2}+1}| = n$. Choose $B_i \subseteq A_i$ with $|B_i| = i$ for $i = 1, \cdots, \frac{n}{2} - 1$. Let $C_i = A_i \setminus B_i$. Then $\{B_1, B_2, \cdots, B_{\frac{n}{2}-1}, C_{\frac{n}{2}-1}, \cdots, C_2, C_1, A_{\frac{n}{2}}, A_{\frac{n}{2}+1}\}$ is a matching decomposition as desired.

Case 2: n is odd.

The proof is similar to that of case 1. ∎

Graph Theory(I) - Final Exam

7. If G is n-connected, $n \geq 2$, then for every n-subset of $V(G)$, there exists a cycle which contains these n vertices.

Proof.

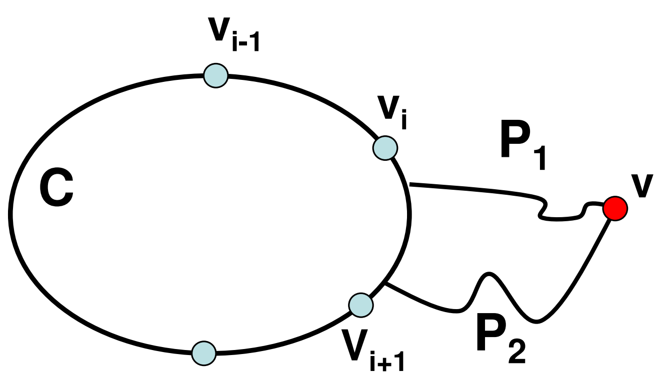
By induction on n.

For $n = 2$, by Menger Theorem, there exist two internally disjoint paths between any two vertices. Hence for the case $n = 2$, the statement is true.

Hypothesis: Suppose the statement is true for $n < k$.

Consider the case $n = k$. Let S be a k-subset of $V(G)$ and $v \in S$. Then $G - v$ is $k - 1$ connected. By the induction hypothesis, there exists a cycle C in $G - v$ containing all vertices in $S - v$.

W.L.O.G, we denote $\{v_1, v_2, \cdots, v_{k-1}\}$ the set of the $k - 1$ vertices in $S - v$ with respect to the order on C. By Menger Theorem, there exist k internally disjoint paths in G from v to C. By pigenhole principle, at least two end-vertices of those paths belong to the same interval $[v_i, v_{i+1}]$ on C for some i. Hence there exists a cycle $C' = \{v - P_1 - C - P_2 - v\}$ containing all vertices in S.



Graph Theory(I) - Final Exam

8. If $|G| \geq 3$ and $\kappa(G) \geq \beta(G)$ where $\beta(G)$ is the independence number of G, then G has a Hamiltonian cycle.

Proof.

Suppose to the contrary. Let $C = \{v_1, v_2, \cdots, v_t\}$ be a longest cycle in G and $t < |G|$.

Let $x \in V(G) \setminus C$. By Menger Theorem, there exists an $x - C$ fan $F = \{P_i | i \in I \subseteq \mathbb{Z}_t, |I| = \kappa(G)\}$, where P_i is an $x - v_i$ path.

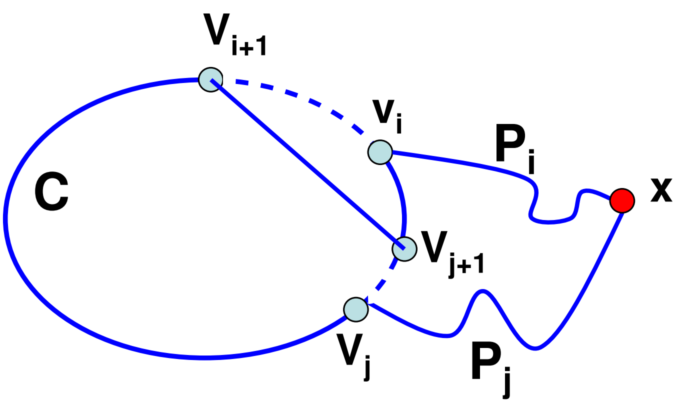
Claim: $\{x\} \cup \{v_{i+1} | i \in I\}$ is an independent set of G.

Then $\beta(G) \geq \kappa(G) + 1$, a contradiction to the assumption $\kappa(G) \geq \beta(G)$. ∎

proof of the claim.

Case 1: $x \not\sim v_{i+1}$. It is easy to see that $x \not\sim v_{i+1}$ for all $i \in I$, for otherwise there exists a longer cycle $x - v_i - C - v_{i+1} - x$.

Case 2: $v_{i+1} \not\sim v_{j+1}$ for all $i, j \in I$. Suppose not. Then $P_i \cup P_j \cup C \cup \{v_{i+1}v_{j+1}\} - \{v_iv_{i+1}, v_jv_{j+1}\}$ yields a longer cycle than C, as shown below. Thus, the set $\{x\} \cup \{v_{i+1} | i \in I\}$ is an independent set of G.



Graph Theory(I) - Final Exam

9. Prove that $\chi(G) \leq \Delta(G)$ except G is an odd cycle or a complete graph.

Proof.

Let $k = \Delta(G)$. We may assume $k \geq 3$, since G is a complete graph when $k = 1$, and G is an odd cycle or is bipartite when $k = 2$, in which case the statement holds.

Our aim is to order the vertices so that each has at most $k-1$ lower-indexed neighbors; hence greedy coloring for such an ordering yields the bound.

When G is not k-regular, we can choose a vertex of degree less than k as v_n. Since G is connected, we can grow a spanning tree from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, v_2, \cdots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower-indexed neighbors, and the greedy coloring uses at most k colors.

In the remaining case, G is k-regular.

Case 1: G has a cut-vertex x.

Let G' be a subgraph consisting of a component of $G-x$ together with its edges to x. The degree of x in G' is less than k, so the method above provides a k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G-x$, we can make the colorings agree on x to complete a proper k-coloring of G.

Case 2: G is 2-connected.

In this case, G must have such a triple v_1, v_2, v_n that v_1, v_2 are incident to v_n, $v_1 \not\leftrightarrow v_2$, and $G - \{v_1, v_2\}$ is still connected(This proof is left to readers). Then we index the vertices of a spanning tree of $G - \{v_1, v_2\}$ using $3, \cdots, n$ such that labels increase along paths to the root v_n. Similarly, each vertex has at most $k-1$ lower-indexed neighbors. Moreover, the greedy coloring also uses at most $k-1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color. ∎

Graph Theory(I) - Final Exam

10. Prove that if D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of G.

Proof.

Let D' be a maximal acyclic subgraph of D. Define $f(v)$ be the length of a longest path in D' ending at v for all $v \in V(G)$.

Our aim is to show f is a proper coloring of G. Note that $f(v) \in \{0, 1, \cdots, l(G)\}$. It is easy to see that $f(y) > f(x)$ if $x \to y$ in D'. Hence f strictly increases along any path in D'.

For all $u \to v \in D$, either $u \to v$ in D' or adding the arc $u \to v$ to D' creates a cycle since D' is a maximal acyclic subgraph of D. Both cases implicate that there is a $u - v$-path in D'. Therefore $f(u) \neq f(v)$.

Now we want to prove that equality holds for some orientation of G. Let f be a proper $\chi(G)$-coloring of G.

For all $uv \in E(G)$, define an orientation D: $u \to v$ if and only if $f(u) < f(v)$.

Then any path is of length at most $\chi(G) - 1$; hence $\chi(G) \geq 1 + l(D)$. (In fact, there exists a path from $f^{-1}(1)$ to $f^{-1}(\chi(G))$.) It implies that the equality holds for some orientation. ∎