Beautiful Results of Graph Theory(I)

1. **Sperner's Lemma**
   Every properly labeled simplicial subdivision has a completely labeled cell.

2. Let rectangle T be titled with rectangles $T_1, T_2, \cdots, T_l$. If each T_i has an integral side, then T has an integral side.

3. $\forall k, g \geq 3$, a $(k; g)$-cage is 2-connected.

4. If G is a 3-connected graph and $|G| > 4$, then G has an edge e such that G/e is 3-connected.

5. Let G be a graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_p$. If for every $1 \leq n \leq p-1$ and $1 \leq k \leq \lfloor (p-n+2)/2 \rfloor$, $d_k \leq k+n-2 \Rightarrow d_{p-n+1} \geq p-k$, then G is n-connected.

6. If G is n-connected, $n \geq 2$, then for every n-subset of $V(G)$, there exists a cycle which contains these n vertices.

7. If $diam(G) = 2$, then $\kappa_1(G) = \delta(G)$.

8. **P. Hall's Theorem**
   A collection $\mathscr{S}$ of n sets $\{S_1, S_2, \cdots, S_n\}$ has an SDR if and only if for every $1 \leq k \leq n$, any collection of k sets in $\mathscr{S}$ contains at least k distinct elements in $\bigcup_{i=1}^{n} S_i$.

9. **Petersen's Theorem**
   Every $2k$-regular graph can be decomposed into k 2-factors.

10. If $|G| \geq 3$ and $\kappa(G) \geq \beta(G)$ where $\beta(G)$ is the independence number of G, then G has a Hamiltonian cycle.

11. Every bipartite graph G has a supergraph which is a bipartite $\Delta(G)$-regular graph.

12. **König's Theorem**
    For every bipartite graph G, $\chi'(G) = \Delta(G)$.

13. **Brook's Theorem**
    $\chi(G) \leq \Delta(G)$ except G is an odd cycle or a complete graph.

14. **5-color Theorem**
    Every planar graph is 5-colorable.

15. **Roy-Gallai's Theorem**
    If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1+l(D)$. Furthermore, equality holds for some orientation of G.

16. **Vizing's Theorem for Simple Graphs**
    $\chi'(G) \leq \Delta(G) + 1$.

17. If G has a k-edge-coloring, then G has an equalized k-coloring.

18. A graph G is bipartite if and only if G contains no odd cycles.

19. A graph G can be decomposed into cycles if and only if every vertex of G is of even degree.

20. For every connected graph G with $\delta(G) \geq 3$, $L^2(G)$ has a Hamiltonian cycle.