Graph Theory (I), Homework 3-1 by 吳政軒

Let $H_i^{\,j}$ be the j-th Hamiltonian cycle when k= i

Let $V(C_p)=\{1,2,...,p\}$, Define $d_i=\{(x,y)\,|\,d(x,y)=i, xy\in V(C_p)\}$

where $d(x,y)$ is the length of shortest path from x to y

(i) k=2

Case1: if p is odd

=> $H_1^{\,2}$ 即爲 d_1 構成的 cycle < 1,2,3,…p-1, p >

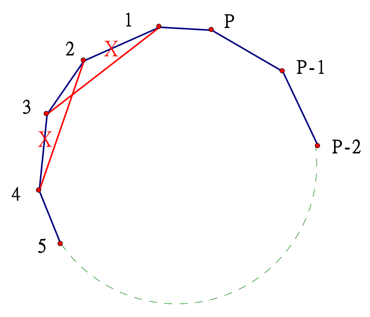
$H_2^{\,2}$ 即爲 d_2 構成的 cycle < 1,3,5,…,p,2,4,…,p-1>

Case1: if p is even

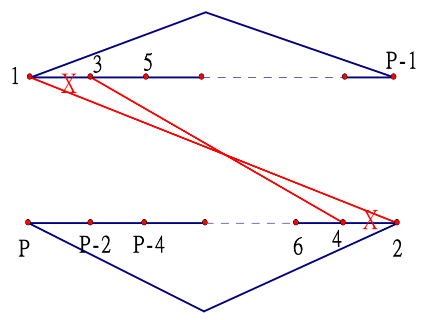
=> $H_1^{\,2}$ 即爲 $d_1-\{(1,2),(3,4)\}+\{(1,3),(2,4)\}$ 構成的 cycle

$H_2^{\,2}$ 即爲 $d_2-\{(1,3),(2,4)\}+\{(1,2),(3,4)\}$ 構成的 cycle

$H_1^{\,2}$



$H_2^{\,2}$



(ii) k=3

Case1: if (p,3)=1, i.e $p\equiv 1 \ or\ 2 \pmod 3$

=> $H_1^{\,3}=H_1^{\,2}$, $H_2^{\,3}=H_2^{\,2}$, $H_3^{\,3}$ 即爲 d_3 構成的 cycle

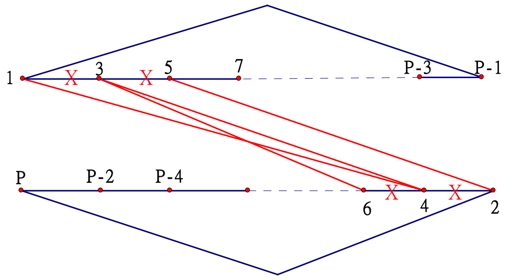
Case2: if $p\equiv 0 \pmod 2$ and $p\equiv 0 \pmod 3$

=> $H_1^{\,3}=H_1^{\,2}$,

$H_2{}^3$即為$d_2-\{(1,3),(2,4),(3,5),(4,6)\}+\{(1,4),(2,5),(3,6),(3,4)\}$構成的 cycle

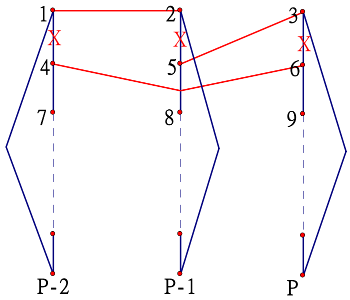
$H_3{}^3$即為$d_3-\{(1,4),(2,5),(3,6)\}+\{(3,5),(4,6),(1,2)\}$構成的 cycle

$H_2{}^3$



<1,4,3,6,8,…,p,2,5,7,…,p-1>

$H_3{}^3$



Case3: if $p \equiv 0 \pmod 3$ *and* $p \equiv 1 \pmod 2$

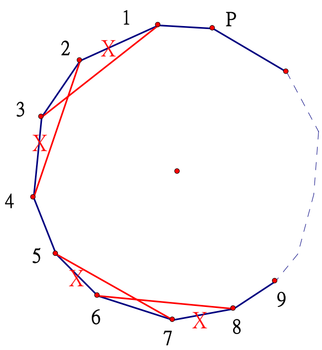
=> $H_1{}^3$即為$d_1-\{(1,2),(3,4),(5,6),(7,8)\}+\{(1,3),(2,4),(5,7),(6,8)\}$構成的 cycle

$H_2{}^3$即為

$d_2-\{(1,3),(2,4),(3,5),(4,6),(5,7),(6,8)\}+\{(1,4),(2,5),(3,6),(3,4),(5,6),(7,8)\}$
構成的 cycle

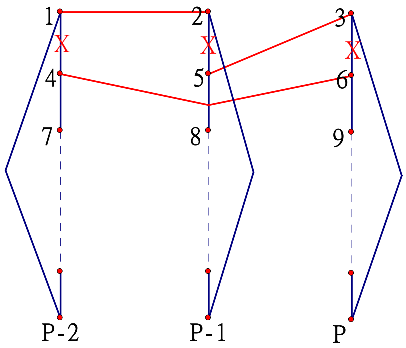
$H_3{}^3$即為$d_3-\{(1,4),(2,5),(3,6)\}+\{(3,5),(4,6),(1,2)\}$構成的 cycle

$H_1{}^3$



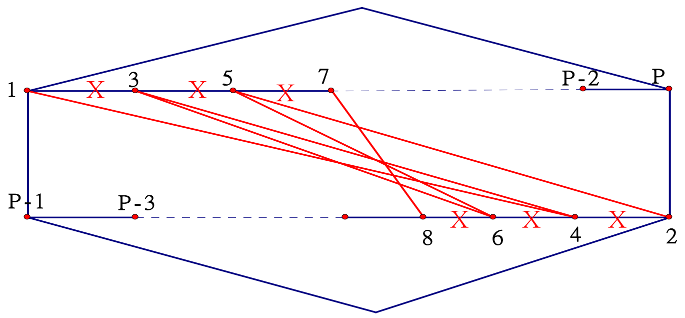
<1,3,2,4,5,7,6,8,9,…p>

$H_3{}^3$



<1,2,p-1,p-4,…,5,3,p,p-3,…,6,4,7,…,p-2>

$H_2^{\,3}$



<1,4,3,6,5,2,p,p-2,…,7,8,10,…,p-3,p-1>

(iii) k=4

Case1: $p \equiv 1,3 \pmod 4$

=> $H_1^{\,4} = H_1^{\,3}$， $H_2^{\,4} = H_2^{\,3}$， $H_3^{\,4} = H_3^{\,3}$， $H_4^{\,4}$爲d_4構成的 cycle

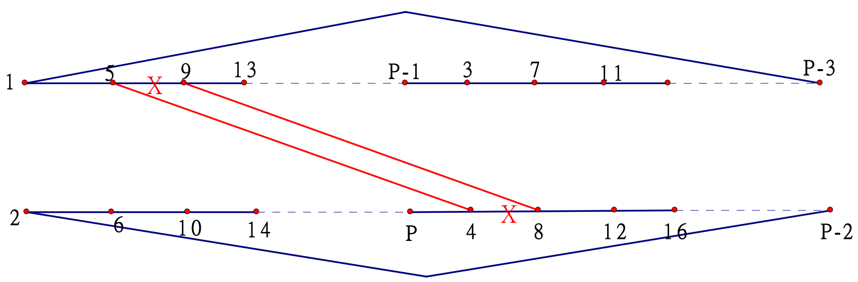
Case2: $p \equiv 2 \pmod 4$

=> $H_1^{\,4}$爲$H_1^{\,3} - \{(4,5),(8,9)\} + \{(4,8),(5,9)\}$構成的 cycle

$H_2^{\,4} = H_2^{\,3}$， $H_3^{\,4} = H_3^{\,3}$

$H_4^{\,4}$爲$d_4 - \{(4,8),(5,9)\} + \{(4,5),(8,9)\}$構成的 cycle

$H_4^{\,4}$



Case3: $p \equiv 0 \pmod 4$

=> $H_1^{\,4}$爲$H_1^{\,3} - \{(4,5),(6,7),(8,9),(9,10)\} + \{(4,8),(5,9),(6,10),(7,9)\}$

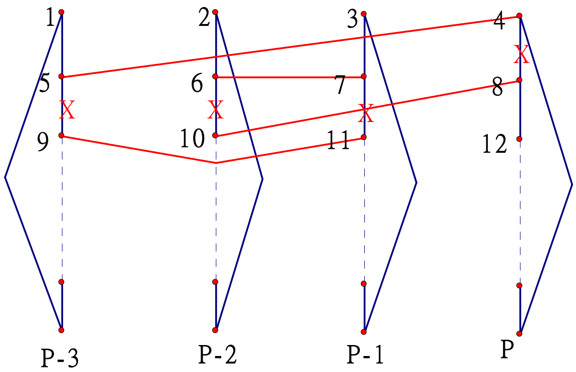
構成的 cycle

$H_2^{\,4}$爲$H_2^{\,3} - \{(7,9),(8,10),(9,11)\} + \{(7,11),(8,9),(9,10)\}$構成的 cycle

$H_4^{\,4}$爲$d_4 - \{(4,8),(5,9),(6,10),(7,11)\} + \{(4,5),(6,7),(8,10),(9,11)\}$構成

的 cycle and $H_3^{\,4} = H_3^{\,3}$

$H_4^{\ 4}$



<1,5,4,p,p-4,…,12,8,10,14,…,p-2,2,6,7,3,p-1,p-4,…,11,9,13,…,p-3>

Graph Theory (I), Homework 3-2 by 曾雅榕

2. Let D be a directed graph such that for each

$v \in V(D)$**,** $\deg_D^+(v) = \deg_D^-(v) = t \geq 1$**.**

Prove that D can be decomposed into t directed 1-factors.(with indegree 1 and outdegree1)

Proof:

Assume D is connected and $V(D) = \{v_1, v_2, v_3, ...v_n\}$

Otherwise, consider each component of D

Since $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v) = t \geq 1$

$\Rightarrow$ D is strong connected.

$\Rightarrow$ D has a directed E.C

Called C: $x_1 \to x_2 \to x_3 \to ..... \to x_q$ where $q = \|D\|$

Define a bipartite graph H with two partite set A,B

Where $A = \{a_1, a_2, a_3, ...a_n\}$, $B = \{b_1, b_2, b_3, ...b_n\}$

$a_{x_i} \leftrightarrow b_{x_j}$ in H $\Leftrightarrow$ $x_i \to x_j$ in C

$\Rightarrow$ H is a t-regular bipartite graph

By König's Theorem

$\Rightarrow$ H has an 1-factor M

$\Rightarrow$ M in H corresponds to a directed 1-factor in D

$\Rightarrow$ D can be decomposed into t 1-factors.

Graph Theory (I), Homework 3-3 by 陳柏澍

3. Prove that G is a perfect graph if and only if $\overline{G}$ is a perfect graph.

Def. We say that G' is obtained from G by expanding the vertex $x \in V(G)$

if and only if $\begin{cases} V(G') = V(G) \cup \{x'\} \\ E(G') = E(G) \cup \{x'v \mid v \in N_G[x]\} \end{cases}$.

Lemma: If G is perfect graph, then G' obtained from expanding a vertex $x \in V(G)$ from G is again a perfect graph.

pf of Lemma:

Prove by induction on $|G|$:
Expanding the vertex of K_1 yields K_2 which is clearly perfect.
Suppose G is a nontrivial perfect graph and G' is obtained from G by expanding the vertex $x \in V(G)$.

Claim: $\forall H \le G'$, χ(H)=ω(H).

Case1. $H < G'$

i. $|V(H) \cap \{x, x'\}| \le 1$

H is isomorphic to an induced subgraph of G.
Such G is perfect, then we are done.

ii. $\{$x,x'$\} \subseteq V(H)$

H is obtained from $H' := H - x'$ where $|H'| < |G|$ by expanding a vertex $x \in V(H')$. Then, by induction hypothesis, done.

Case2. $H = G'$

Claim: $\chi(G') \le \omega(G')$

Let $\omega(G) := \omega$.
Observe that $\omega(G') \in \{\omega, \omega+1\}$.
Case2.1. $\omega(G')$=ω+1

$\chi(G') \le \chi(G)+1 = \omega(G)+1 = \omega+1 = \omega(G')$

Clearly, $\chi(G') \ge \omega(G')$.

$\therefore\ \chi(G') = \omega(G')$.

Case2.2. $\omega(G') = \omega$

Note, x lines in no $K_\omega \subseteq G$, otherwise $\omega(G') = \omega+1$.

Coloring G with ω colors.

Let X be a color class containing x.

Then, $\begin{cases} |X \cap V(K_\omega)| = 1 \\ x \notin X \cap V(K_\omega) \end{cases}$ for any $K_\omega \le G$.

$\left(\text{Clear, } |X \cap V(K_\omega)| \le 1. \text{ If } X \cap V(K_\omega) = \phi, \text{ then } \omega \text{ colors will not eonugh to color } G. \right)$

Let $\tilde{G} = G - (X \setminus \{x\})$.

Thus, $\chi(\tilde{G}) = \omega - 1$.

$\because V(G') \setminus V(\tilde{G}) = (X \setminus \{x\}) \cup \{x'\}$ is an independent set.

$\therefore\ \chi(G') \le (\omega-1)+1 = \omega$.

Thus, $\chi(G') = \omega(G')$. ($\because \chi(G') \ge \omega(G')$)

pf:

It suffices to show that if G is perfect then $\overline{G}$ is also perfect.

Applying induction on $|G|$:

$|G| = 1$ is clear.

Suppose it holds for $|G| \ge 2$.

It suffices to show that $\chi(\overline{G}) \le \omega(\overline{G})$.

($\because$ Every proper induced subgraph of $\overline{G}$ is the complement of a proper induced subgraph of G.

$\therefore$ By induction hypothesis, every proper induced subgraph of $\overline{G}$ is perfect.)

Let $\mathcal{K} = \{ S \subseteq V(G) \mid \langle S \rangle_G = \text{ maximal } K_t \text{ for some } t \le \omega(G) \}$

$\mathcal{A} = \{ A \subseteq V(G) \mid A \text{ is an independent set s.t. } |A| = \alpha(G) \}$.

Claim: $\exists K \in \mathcal{K}$ s.t. $K \cap A = \phi$ for all $A \in \mathcal{A}$.

pf of claim:

Suppose not.

i.e. $\forall K \in \mathcal{K}\ \exists A_K \in \mathcal{A}$ s.t. $K \cap A_K = \phi$.

Constructing G' by replacing each vertex x of G by a complete graph G_x of order $k(x) := \left|\left\{K \in \mathcal{K} \,\middle|\, x \in A_K\right\}\right|$ and joining all the vertices of G_x to all the vertices of G_y if x and y are adjacent in G.

Observe that G' can be obtained by repeating vertex expansion= from the graph $G\left[\left\{x \in V(G) \,\middle|\, k(x) > 0\right\}\right]$ which is perfect since G is perfect.

By the Lemma, G' is perfect.

$\therefore \chi(G') = \omega(G')$. $\therefore \chi(G') \le \omega(G')$.

Observe $\omega(G') = \sum_{x \in \mathcal{X}} k(x)$ where $\mathcal{X} \in \mathcal{K}$

$$= \left|\left\{(x, K) \,\middle|\, x \in \mathcal{X}, K \in \mathcal{K}, x \in A_K\right\}\right|$$

$$= \sum_{K \in \mathcal{K}} \left|\mathcal{X} \cap A_K\right| \le \left|\mathcal{K}\right| - 1.$$

$$\left|G'\right| = \sum_{x \in V(G')} k(x)$$

$$= \left|\left\{(x, K) \,\middle|\, x \in V(G), K \in \mathcal{K}, x \in A_K\right\}\right|$$

$$= \sum_{K \in \mathcal{K}} \left|A_K\right|$$

$$= \left|\mathcal{K}\right| \cdot \alpha(G).$$

$\because \alpha(G') \le \alpha(G)$.

$$\therefore \chi(G') \ge \frac{\left|G'\right|}{\alpha(G')} \ge \frac{\left|G'\right|}{\alpha(G)} = \left|\mathcal{K}\right|.$$

Thus, $\chi(G') \ge \left|\mathcal{K}\right| > \left|\mathcal{K}\right| - 1 \ge \omega(G')$. A contradiction.

Thus, $\omega(\overline{G} - K) = \alpha(G - K) < \alpha(G) = \omega(\overline{G})$.

$\therefore \chi(\overline{G}) \le \chi(\overline{G} - K) + 1 = \omega(\overline{G} - K) + 1 \le \omega(\overline{G})$.

Graph Theory (I), Homework 3-4 by 張澍仁

1° Let $k = \chi'(G)$

If $k \le \frac{3}{2}\Delta(G)$, then done.

Assume that $k > \frac{3}{2}\Delta(G)$.

2° $\because \chi'(G) = k$

$\therefore \exists G'$ be a minimal subgraph of G

s.t. $\chi'(G') = k$ and $\chi'(G'-e) = k-1 \ \forall e \in E(G')$.

3° Let $x, y \in V(G')$ s.t. there are $\mu(G')$ edges between x, y.

Let e be an edge of them.

4° Consider the graph : $G' - e$

We found that:

1) $\chi'(G'-e) = k-1$

2) all x,y 所連的邊(in $G' - e$)一定要用完 k-1 色.

(else by 1) ,接回 e 並著未用到的顏色 => $\chi'(G') = k-1 \quad \rightarrow\leftarrow$)

3) $\because d_{G'-e}(x) = d_{G'}(x) - 1 \le \Delta(G) - 1$.

$\therefore$ x 最多用到$\Delta(G)-1$色 (其中與 y 共用$\mu(G')-1$色).

=> x 至少還有$(k-1)-(\Delta(G)-1) = k-(\Delta(G))$色未用到.

=> by 2) , y 一定要用到這些顏色(這些顏色不在 x,y 之間).

同理,x 亦至少用到$k-(\Delta(G))$色,且這些顏色不在 x,y 之間.

=> $k-1 \ge (\mu(G')-1) + 2(k-\Delta(G))$. $\cdots$(*)

4° By Vizing Theorem

$$\Rightarrow k = \chi'(G) = \chi'(G') \le \Delta(G') + \mu(G') \le \Delta(G) + \mu(G')$$

$$\Rightarrow \mu(G') \ge k - \Delta(G)$$

$$\therefore (*) \Rightarrow k - 1 \ge (k - \Delta(G) - 1) + 2(k - \Delta(G))$$

$$\Rightarrow k \le \frac{3}{2}\Delta(G). \qquad \rightarrow\leftarrow$$

□

Graph Theory (I), Homework 3-5 by 陳宏賓

6. Prove that if $\|G\| = \binom{n+1}{2}$ and $\Delta(G) \leq \lfloor n/2 \rfloor$, then G can be decomposed into n subgraphs $G_1, G_2, \cdots, G_n$ such that G_i is induced by a matching with i edges for all i.

Lemma 1: Every k-edge colorable graph has an equitable k-edge coloring.

Proof.

Case 1: n is even.

By Vizing's Theorem and Lemma 1, G has an equitable $(\frac{n}{2}+1)$-edge coloring with color classes $A_1, A_2, \cdots, A_{\frac{n}{2}+1}$ s.t. $|A_i| = n-1$ for $i = 1, \cdots, \frac{n}{2}$ and $|A_{\frac{n}{2}+1}| = n$. Choose $B_i \subseteq A_i$ with $|B_i| = i$ for $i = 1, \cdots, \frac{n}{2} - 1$. Let $C_i = A_i \setminus B_i$. Then $\{B_1, B_2, \cdots, B_{\frac{n}{2}-1}, C_{\frac{n}{2}-1}, \cdots, C_2, C_1, A_{\frac{n}{2}}, A_{\frac{n}{2}+1}\}$ is a matching decomposition as desired.

Case 2: n is odd.

The proof is similar to that of case 1. ∎

Graph Theory (I), Homework 3-6 by 張雁婷

6. Let $\boldsymbol{D}$ be an $\boldsymbol{n}$-regular digraph of order $2\boldsymbol{n}+1$, $\boldsymbol{n} \geq 1$. Prove it or disprove that $\boldsymbol{D}$ has a directed Hamiltonian cycle. ($\boldsymbol{D}$ is $\boldsymbol{n}$-regular if $\forall \boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{D})$, $\boldsymbol{d}^{+}(\boldsymbol{v}) = \boldsymbol{d}^{-}(\boldsymbol{v}) = \boldsymbol{n}$.)

Pf.

Note. The underlying graph of $\boldsymbol{D}$ is $\boldsymbol{K}_{2\boldsymbol{n}+1}$, and hence $\boldsymbol{D}$ is a tournament of order $2\boldsymbol{n}+1$.

Claim: $\boldsymbol{D}$ has a directed Hamiltonian cycle.

Pf.

Suppose not, let $\boldsymbol{C}$ be the maximum cycle in $\boldsymbol{D}$, and $|\boldsymbol{V}(\boldsymbol{C})| \leq 2\boldsymbol{n}$. Let $\boldsymbol{u}$ be a vertex not in $\boldsymbol{C}$. Since $\boldsymbol{D}$ is a tournament, there is a directed edge between $\boldsymbol{u}$ and each vertex in $\boldsymbol{C}$.

Case (i)

If there are two edges between $\boldsymbol{u}$ and $\boldsymbol{C}$ of different directions, we can find two adjoining vertices $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{C}$ such that the directions from $\boldsymbol{u}$ to $\boldsymbol{x}$ and from $\boldsymbol{u}$ to $\boldsymbol{y}$ are different. Hence we have a cycle larger than $\boldsymbol{C}$, a contradiction.

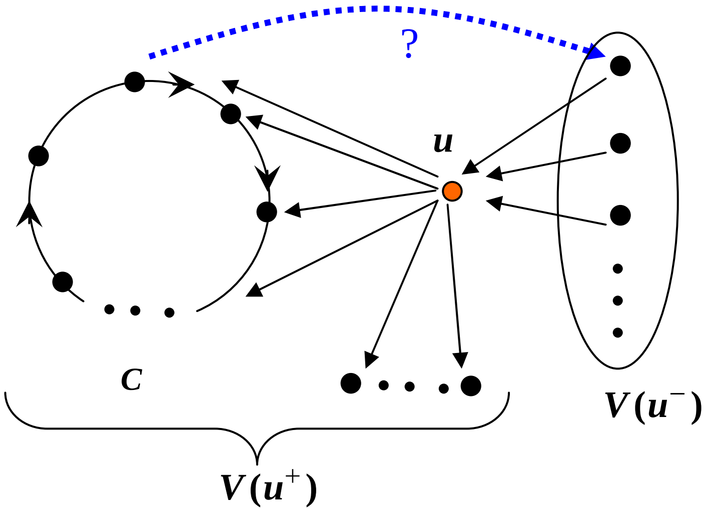
Case (ii) the directions from $\boldsymbol{u}$ to each vertex in $\boldsymbol{C}$ are the same.

$$\because \boldsymbol{d}^{+}(\boldsymbol{u}) = \boldsymbol{d}^{-}(\boldsymbol{u}) = \boldsymbol{n} \quad \therefore |\boldsymbol{V}(\boldsymbol{C})| \leq \boldsymbol{n}$$

Case (ii)-1 Suppose the direction between $\boldsymbol{u}$ and $\boldsymbol{C}$ are all $\boldsymbol{u} \to \boldsymbol{C}$.

Let $\boldsymbol{V}(\boldsymbol{u}^{-}) = \{\boldsymbol{z} \in \boldsymbol{V}(\boldsymbol{D}) \mid \boldsymbol{z} \to \boldsymbol{u} \text{ in } \boldsymbol{D}\}$ and $\boldsymbol{V}(\boldsymbol{u}^{+}) = \{\boldsymbol{z} \in \boldsymbol{V}(\boldsymbol{D}) \mid \boldsymbol{z} \leftarrow \boldsymbol{u} \text{ in } \boldsymbol{D}\}$.

Note that $|\boldsymbol{V}(\boldsymbol{u}^{-})| = |\boldsymbol{V}(\boldsymbol{u}^{+})|$. For each vertex $\boldsymbol{z} \in \boldsymbol{V}(\boldsymbol{u}^{-})$, $\boldsymbol{z}$ is also adjacent to each vertex in $\boldsymbol{C}$, since $\boldsymbol{D}$ is a tournament. If there is a directed edge form $\boldsymbol{C}$ to $\boldsymbol{z}$, then we have a larger cycle through $\boldsymbol{u}$, a contradiction. Hence the direction between **C** and $\boldsymbol{z}$ are all $\boldsymbol{z} \to \boldsymbol{C}$. And $\because |\boldsymbol{V}(\boldsymbol{u}^{-}) + \{\boldsymbol{u}\}| = \boldsymbol{n}+1$ $\therefore$ for each vertex $x \in C$, $\mathbf{deg}^{-}(x) \geq n+1$, it contradicts to that D is n-regular. Therefore, D has a directed Hamiltonian cycle.



Case (ii)-2 Similar discussion for the directions between u and C are all $C \rightarrow u$.

Graph Theory (I), Homework 3-7 by 羅元勳

7. Let $n = k(2l+1)$. Construct a non-Hamiltonian complete k-partite graph with n vertices and minimum degree $\frac{n}{2}\frac{k-1}{k}\frac{2l}{2l+1}$.

Sol.

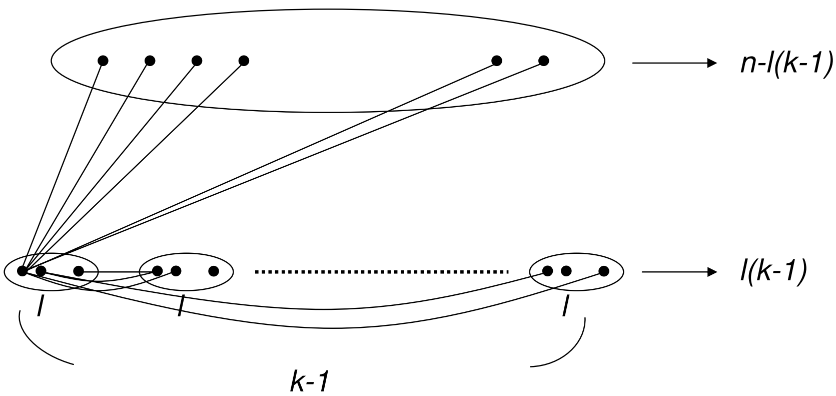
Minimum degree $= \frac{k(2l+1)}{2}\frac{k-1}{k}\frac{2l}{2l+1} = l(k-1)$.

Define $G = K_{n-l(k-1),l,l,\cdots,l}$, which is a complete k-partite graph.

Claim : G has no Hamiltonian cycle.

Proof. $n - l(k-1) = 2kl + k - kl + l$
$= kl + k + l = l(k+1) + k > l(k-1)$

G :



Thus, the vertex number of the first partite set is larger than the sum of the others, then it's easy to see that G has no Hamiltonian cycles.

Graph Theory (I), Homework 3-8 by 黃皓文

8. Dirac proved that every 2-connected simple graph G has a cycle of length at least $\min\{n(G), 2\delta(G)\}$**. Use this to prove that every 2k-regular graph with 4k+1 vertices is Hamiltonian.**

Proof: Let G be a 2k-regular graph with 4k+1 vertices
Claim: G is 2-connected

Pf of claim: Suppose not. Then there is a vertex $u \in V(G)$ s.t. $G-u$ is not connected. Let $G_1,\dots,G_t$ be components of $G-u$. Since G is 2k-regular. $|G_i| \geq 2k$ for each *i* and $|G| = \sum_{i=1}^{t}|G_i| + 1 \geq 2tk+1$

$\Rightarrow 4k+1 \geq 2tk+1 \Rightarrow t \leq 2$

Hence t=2 and $|G_1| = |G_2| = 2k$.

As $\deg_G(u) = 2k$, there is one vertex $v_1 \in V(G_1)$ s.t. $\deg_G(v_1) = 2k$

$\Rightarrow |G_1| \geq 2k+1$, a contradiction.

Thus, G is 2-connected

By the result of Dirac, G has a cycle of length at least min{4k+1,4k}=4k
Let C be a longest cycle in G.
If |C|=4k+1, then C is a Hamiltonian cycle of G.

If |C|=4k, say $C: x_1 - y_1 - x_2 - y_2 - \dots - x_{2k} - y_{2k}$, there is only one vertex $z \in V(G-C)$. By maximality of C and $\deg_G(z) = 2k$, we see either $N_G(z) = \{x_1,\dots x_{2k}\}$ or $N_G(z) = \{y_1,\dots y_{2k}\}$.W.L.O.G. assume $N_G(z) = \{x_1,\dots x_{2k}\}$

Claim: $y_i \sim y_j$ for some i,j

Pf of claim: Suppose not. i.e. $\forall i \neq j \in \{1,\dots,2k\}$ y_i is not adjacent to y_j.

Then $N_G(y_i) \subseteq \{x_1, \ldots x_{2k}\}$ for each i.

As $\deg_G(y_i) = 2k$, $N_G(y_i) = \{x_1, \ldots x_{2k}\}$ for each i.

$\Rightarrow \{z, y_1, y_2, ..., y_{2k}\} \subseteq N_G(x_1)$,a contradiction.

By claim, there is a longer cycle

$z - x_j - y_{j-1} - x_{i-1} - .... - y_i - y_j - x_{j+1} - ... - x_i - z$, a contradiction.

Graph Theory (I), Homework 3-9 by 黃志文

9.

(a) $G\Box K_2 = G\cup G'\cup\{v_1v'_1, v_2v'_2, v_3v'_3, \cdots, v_nv'_n\}$,where G' is a copy of G and $V(G)=\{v_1,v_2,v_3,\cdots,v_n\}$ and $V(G')=\{v'_1,v'_2,v'_3,\cdots,v'_n\}$.

Case 1: $\chi'(G)=\triangle(G)$

Let $v_i'v_j'$ be colored the same color of v_iv_j and $v_iv'_i$ be colored a new color.

Then $\chi'(G\Box K_2)=\triangle(G)+1=\triangle(G\Box K_2)$.

Case 2: $\chi'(G)=\triangle(G)+1$

Let $v_i'v_j'$ be colored the same color of v_iv_j . Since $\chi'(G)=\triangle(G)+1$,there is a color doesn't appear near v_i. Thus color $v_iv'_i$ with the color.

Then $\chi'(G\Box K_2)=\triangle(G)+1=\triangle(G\Box K_2)$.

(b) It is obviously $V((G_1\cup G_2)\Box(H_1\cup H_2))=V((G_1\Box H_2)\cup(G_2\Box H_1))$ and let the vertex set is $\{(v,w)\mid v\in V, w\in W\}$.

1° $\forall$ $e=\{(v,w),(v',w')\}\in E((G_1\cup G_2)\Box(H_1\cup H_2))$.

(i) $v=v'$ and ($ww'\in E(H_1)$ or $ww'\in E(H_2)$).

$\Rightarrow$($v=v'$ and $ww'\in E(H_1)$) or ($v=v'$ and $ww'\in E(H_2)$).

$\Rightarrow e\in(G_1\Box H_2)\cup(G_2\Box H_1)$.

(ii) $w=w'$ and ($vv'\in E(G_1)$ or $vv'\in E(G_2)$).

$\Rightarrow$($w=w'$ and $vv'\in E(G_1)$) or ($w=w'$ and $vv'\in E(G_2)$).

$\Rightarrow e\in(G_1\Box H_2)\cup(G_2\Box H_1)$.

Hence $E((G_1\cup G_2)\Box(H_1\cup H_2))\subseteq E((G_1\Box H_2)\cup(G_2\Box H_1))$.

2° $\forall$ $e\in E(G_1\Box H_2)$ $\Rightarrow$ $e\in E((G_1\cup G_2)\Box(H_1\cup H_2))$.

$\forall$ $e\in E(G_2\Box H_1)$ $\Rightarrow$ $e\in E((G_1\cup G_2)\Box(H_1\cup H_2))$.

Hence $E((G_1\Box H_2)\cup(G_2\Box H_1))\subseteq E((G_1\cup G_2)\Box(H_1\cup H_2))$

So, $E((G_1\Box H_2)\cup(G_2\Box H_1))=E((G_1\cup G_2)\Box(H_1\cup H_2))$.

(c)

1° $\forall$ $(v,w)\in V(G\Box H)$

$\{(v,w),(v,w')\}\in E(G\Box H)$ iff $w'\in N_H(w)$.

$\{(v,w),(v',w)\}\in E(G\Box H)$ iff $v'\in N_G(v)$.

Hence $\deg((v,w))=\deg(v)+\deg(w)$.

Thus $\triangle(G\Box H)=\triangle(G)+\triangle(H)$.

2° Since both G and H are 1-factors ,let $|G|=2n$, $|H|=2m$ and $G=G'\cup nK_2$, $H=H'\cup mK_2$.

Then $G\Box H=(G'\cup nK_2)\Box(H'\cup mK_2)$

$=(G'\Box mK_2)\cup(H'\Box nK_2)$ by (b)

$=(m(G'\Box K_2))\cup(n(H'\Box K_2))$

And $\chi'(G'\square K_2) = \triangle(G'\square K_2) = \triangle(G)$

$\chi'(H'\square K_2) = \triangle(H'\square K_2) = \triangle(H)$ by (a).

Then

$$\begin{aligned}\chi'(G\square H) &= \chi'((m(G'\square K_2)) \cup (n(H'\square K_2))) \\ &\leq \chi'(m(G'\square K_2)) + \chi'(n(H'\square K_2)) \\ &= \chi'(G'\square K_2) + \chi'(H'\square K_2) \\ &= \triangle(G) + \triangle(H) \\ &= \triangle(G\square H) \\ &\leq \chi'(G\square H).\end{aligned}$$

Thus $\chi'(G\square H) = \triangle(G\square H)$.

Graph Theory (I), Homework 3-10 by 張惠蘭

10. Prove that if G is a multigraph with multiplicity μ, then $\chi'(G) \le \Delta(G) + \mu$.

$\Delta = \Delta(G), \quad \mu = \mu(G)$

1. Color G with $\Delta + \mu$ colors as many edges as possible.
   Denote such coloring as f.
2. Consider some edges left uncolored.
   Let $B(v)$ be the set of colors missing at v under f.
   It is clear that $|B(v)| \ge \mu$ for any v.
3. Let uv be and edge which is not colored.

Claim: We can extend the coloring to uv by recoloring some edges colored under f.

Proof.
(We say that "we can have the extension" if we can extend the coloring to uv)

Let $v = v_0$.
Let $a_0 \in B(u)$, $a_1 \in B(v_0)$ and delete a_1 from $B(v_0)$.
Assume $a_1 \notin B(u)$, i.e., $\exists v_1$ s.t. $f(uv_1) = a_1$, otherwise, we can have the extension.

Let $a_2 \in B(v_1)$ and delete a_2 from $B(v_1)$.
Similarity, assume $\exists v_2$ s.t. $f(uv_2) = a_2$.

Let $a_3 \in B(v_2)$ and delete a_3 from $B(v_2)$.
Similarity, assume $\exists v_3$ s.t. $f(uv_3) = a_3$.
Notice that v_3 and v_1 are not necessarily distinct. (see fig 1)

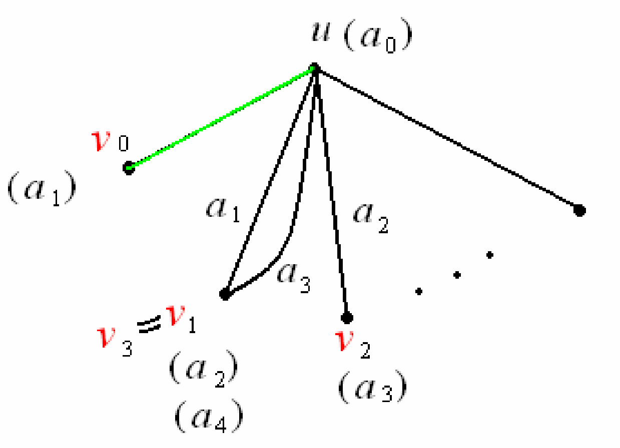


Figure 1.

But, we can always take a color from $B(v)$ in this process, since $|B(v)| \ge \mu$.
(This is different from simple graph)

Now, continue this process and then meet one of the following two cases:

[1] Assume there exists l s.t. $l+1$ is the smallest index s.t. $a_l \in B(v_l)$ and a_l is deleted from some $B(v_{k-1})$, i.e., $a_{l-1} = a_k$.

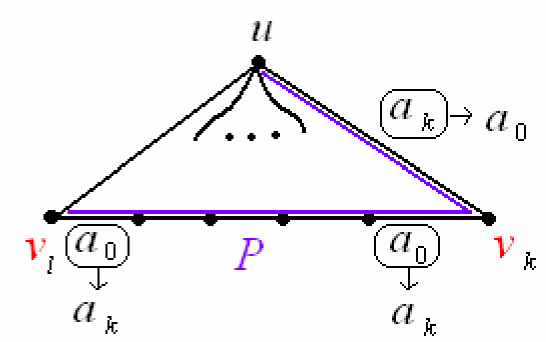
Since a_k is deleted from $B(v_{k-1})$ before we find a_{l-1} in $B(v_l)$, $v_{k-1} \neq v_l$.
Let P be the maximal alternating path of edges colored a_0 and a_k that begins at v_l along color a_0. Such path is well-defined, since each vertex has at most one incident edge in each color.

假設我們可以把uv_t的顏色從a_t改成a_0 properly, 且不改變$B(v_i)$ for any $i<t$，則把uv_i塗成a_{i+1} for $i=0,\cdots,t-1$仍可維持 proper coloring，因爲原本$a_{i+1} \in B(v_i)$.-------------$\otimes$

Consider the following cases:

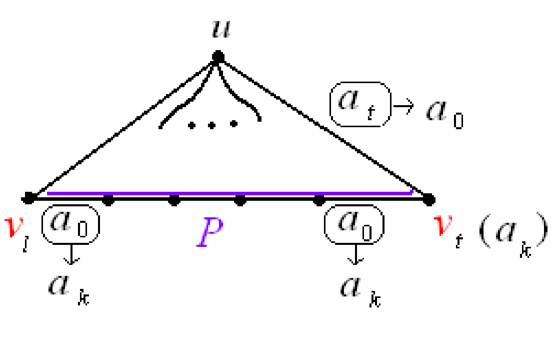
(1) $V(P) \cap \{v_0, \cdots, v_{l-1}\} = \phi$.
Interchange colors a_0 and a_k on P and interchange a_0 and a_l on uv_l.
It is clear that such coloring makes uv_l satisfying the condition of $\otimes$.

(2) $V(P) \cap \{v_0, \cdots, v_{l-1}\} \neq \phi$
Let w be another endpoint of P.
Then there are only three sub-cases:



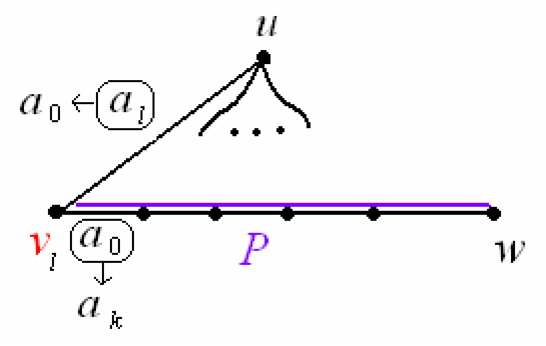
(i) $w = u$

Interchange colors a_0 and a_k on P. Such coloring makes uv_k satisfying the condition of $\otimes$.



(ii) $w = v_t$ for some t and $a_k \in B(v_t)$.

Interchange colors a_0 and a_k on P and change the color of an edge colored a_t with u and v_t as its endpoints to a_0. Such coloring makes uv_t satisfying the condition of $\otimes$.



(iii) $w \notin \{u, v_0, \cdots, v_l\}$

Interchange colors a_0 and a_k on P and change the color of an edge colored a_l with u and v_l as its endpoints to a_0.

Such coloring makes uv_l satisfying the condition of $\otimes$.

Thus if such l exist, we can always extend the coloring to uv

[2] Assume that such l doesn't exist.

Since u has at most Δ incident edges, the process can stop at some v_t s.t. $a_{t+1} \in B(u)$.

Interchange colors a_t and a_{t+1} an edge with endpoints as u and v_t.

Such coloring makes uv_t satisfying the condition of $\otimes$.

Hence we can also extend the coloring to uv.