Graph Theory(I), Homework 2-1 by 連敏筠

(1st Proof)

By induction on ||G|| and it is true for ||G||=0. Let $xy \in E(G)$. Observe that if G has no k disjoint paths, then G/xy has no k disjoint path either. G/xy has an A-B separator Y with fewer than k vertices. In fact, |Y|=k-1. Note that $v_{xy} \in Y$

(otherwise, Y separator A from B in G). Let $X = (Y \setminus \{v_{xy}\}) \bigcup \{x, y\}$. Then |X|=k

and X separates A from B in G.. Now every A-X separator S is also an A-B separator in G.. Hence $|S| \geq k$ and there are k-disjoint A-X paths. Similarly, there are k disjoint X-B paths. Combining the above path we have k vertex-disjoint paths.

(2nd proof) {a}-{b} version

Consider $ab \notin E(G)$.

(Fact) If a,b are in distinct components, done. W.L.O.G. let G be connected min|S|=k.

k=1 is true. Let $S_n(a,b)$ denote the statement that the minimum separator is of size n. Assume our claim is not true. Then there is a smallest integer $m \geq 2$ s.t $S_m(a,b)$ is true but there are fewer than m a-b-disjoint paths.

(*) Among all such graphs of minimum order, let H be the one with minimum size. Then H has the following properties.

(1) If $v_1v_2 \in E(G - \{a,b\})$ then there exist a set U s.t |U|=m-1 and $U \bigcup \{v_i\}$ is an a-b separator for i =1, 2.

(2) For any $w \notin \{a,b\}$ in H not both aw, bw are in H.

(3) If $\{w_1,...,w_m\}$ is an a-b separator, then either $\{aw_i \mid i \in \{1,2,...,m\}\} \subseteq E(H)$

or $\{bw_i \mid i \in \{1,2,...,m\}\} \subseteq E(H)$.

Let P be a shortest a-b path $\| p \| \geq 3$, $P = < a, u_1, u_2, ..., b >$ $u_1u_2 \in E(H)$ $u_1 \neq u_2$.

By (1), $\exists |U| = m-1$ $U \bigcup \{u_i\}$ $i = 1,2$ are a-b separator. Since

$au_1 \in E(H)$ but $bu_1 \notin E(H)$ by (3) $\{u_1\} \bigcup U \subseteq N_H(a)$.

$\Rightarrow$ no vertex in $\{u_1\} \bigcup U$ is adjacent to b (By (2)).

$\Rightarrow \{u_2\} \bigcup U \not\subset N_H(b) \Rightarrow \{u_2\} \bigcup U \subseteq N_H(a) \Rightarrow au_2 \in E(H)$ 矛盾. (since P is the shortest)

Graph Theory(I), Homework 2-2 by 羅元勳

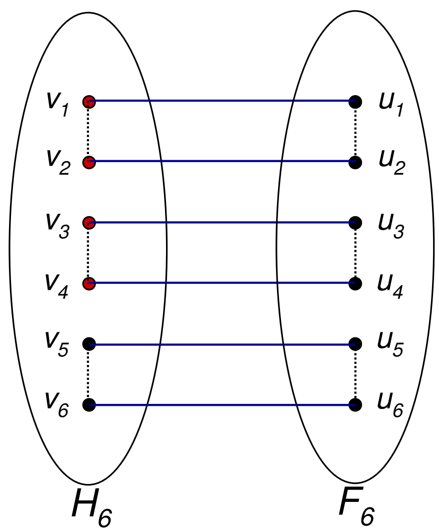
Find a graph with (1) $\kappa(G) > l(G) > 0$ and (2)$\kappa(G) - l(G)$ is as large as powwible.

<u>Sol.</u>

1. K_{2n} is $(2n-1)$-connected and n-linked.
   Then $\kappa(K_{2n}) - l(K_{2n}) = n - 1$.

2. K_{2n+1} is $2n$-connected and n-linked.
   Then $\kappa(K_{2n+1}) - l(K_{2n+1}) = n$.

3. Let $H_{2n} = K_{2n} - \{v_{2i-1}v_{2i} |\ i = 1, 2, \cdots, n\}$ where $V(K_{2n}) = \{v_1, v_2, \cdots, v_{2n}\}$. And let F_{2n} be a copy of H_{2n} with vertex set $\{u_1, u_2, \cdots, u_{2n}\}$.
   Define $G_{2n} = H_{2n} \cup F_{2n} + \{v_i u_i |\ i = 1, 2, \cdots, 2n\}$.
   Example. $n = 3$



   Clearly, $\kappa(G_6) = 5$. Pick $\{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}$, we can't find 3 disjoint paths join v_1 to v_2, v_3 to v_4, and v_5 to v_6, respectively. Then $l(G_6) = 2$.
   Hence $\kappa(G_6) - l(G_6) = 3$.

For general n, we want to count: $\kappa(G_{2n}) - l(G_{2n})$

(a) $\kappa(G_{2n}) = 2n - 1$.

(b) Observe that:

n	l
$1 \sim 9$	$n-1$
$10 \sim 18$	$n-2$
$19 \sim 27$	$n-3$

Then $l(G_{2n}) = n - \lceil \frac{n}{9} \rceil$.(Omit the Proof.)

Hence $\kappa(G_{2n}) - l(G_{2n}) = 2n - 1 - (n - \lceil \frac{n}{9} \rceil) = n - 1 + \lceil \frac{n}{9} \rceil$

■

Graph Theory(I), Homework 2-3 by 陳宏賓

PROBLEM.

Let $k \geq 2$. Show that every k-linked graph of order at least $4k$ contains a cycle of length at least $4k-2$.

Proof.

Let G be a k-linked graph of order at least $4k$. Clearly, G is $2k-1$ connected.

Suppose $C = \{v_1, v_2, \cdots, v_l\}$ is the longest cycle in G with length l less than $4k-2$. Let v be a vertex in $V(G) \setminus C$. Then, there exist at least either $2k-1$ or as many as $|C| = l$ internally disjoint paths from v to C since G is $2k-1$ connected. By the pigeonhole principle, there are two endpoints v_i, v_j of these paths such that v_i and v_j are adjacent in C for some i, j. Therefore, we can extend the original cycle C to a longer cycle $C' = \{v_1, \cdots, v_i, v, v_j, \cdots, v_l\}$ by adding the vertex v, a contradiction to the assumption that C is the longest cycle.

Graph Theory (I), Homework 2-4　by 裴若宇

4. For each graph G of order p, prove that $\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = p$.

$\alpha(G)$：minimum size of vertex cover
$\beta(G)$：maximum size of independent set
$\alpha_1(G)$：minimum size of edge cover
$\beta_1(G)$：maximum size of matching

[proof]
1° claim $\alpha(G) + \beta(G) = p$.

Let A be a maximal independent set of V(G) $\Rightarrow$ $|A| = \beta(G)$
Let $B = V(G) - A$.
$\forall uv \in E(G)$, u 與 v 至少有一點在 B 中
$\Rightarrow$ B 爲 G 的一個 vertex cover. (因爲所有的邊都至少有一點在 B 裡面)
$\Rightarrow |B| \geq \alpha(G)$
$\therefore p = |A| + |B| \geq \alpha(G) + \beta(G)$ …… (♥)

Let C be a minimum vertex cover $\Rightarrow$ $|C| = \alpha(G)$
Let $D = V(G) - C$
$\Rightarrow$ D 中沒有邊了!! (因爲 D 中的每個點都不相鄰)
$\Rightarrow$ D 爲 G 的一個 independent set.
$\Rightarrow |D| \leq \beta(G)$
$\therefore p = |C| + |D| \leq \alpha(G) + \beta(G)$ …… (♥♥)

由(♥)與(♥♥)可知, $\alpha(G) + \beta(G) = p$.

2° claim $\alpha_1(G) + \beta_1(G) = p$.

Let A be a maximum matching $\Rightarrow$ $|A| = \beta_1(G)$
Let $B = A + \{ uv \mid u \in V(G) - V(A), v \in N_G(u) \}$
$\Rightarrow$ B 爲 G 中的一個 edge cover. (因爲 B 把 G 中所有的點都 cover 住了)
$\Rightarrow |B| \geq \alpha_1(G)$
又 A 爲 matching, B 爲 matching + 沒用到的點
故 $p = |A| + |B|$ (因爲 matching 中一個邊用到兩個點)
$\therefore p = |A| + |B| \geq \alpha_1(G) + \beta_1(G)$ …… (♣)

Let C be a minimum edge cover $\Rightarrow$ $|C| = \alpha_1(G)$

$\Rightarrow C$ 中的每個 component 都是 star

設 C 有 k 個 components

在每個 star 中取一個邊, 就可造出一個 matching D

$\Rightarrow |D| \le \beta_1(G)$, and $|D| = k$.

又 $p = k + |C|$ (k 表示 star 的中心點, $|C|$ 代表 star 除了中心點外的點數)

$\therefore p = |C| + |D| \le \alpha_1(G) + \beta_1(G)$ …… (♣♣)

由(♣)與(♣♣)可知, $\alpha_1(G) + \beta_1(G) = p$.

3° 由 1°與 2°知, $\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = p$. (◕‿◕)

Graph Theory(I), Homework 2-5 by 張惠蘭

Fact: *G is 3-connected iff G is either a wheel graph W_n, $n \geq 3$, or obtained by the following two operations from a wheel.*

(1) Adding a new edge,

(2) Splitting a vertex v into v_1 and v_2 s.t. $w \sim v_1$ or $w \sim v_2$ in G for any $w \in N_G(v)$, $v_1 \sim v_2$ in G and $deg_G(v_i) \geq 3$ for $i = 1, 2$.

Claim: If G, 3-connected, has two nonincident edges e_1, e_2 s.t. $G \cdot e_i$ is 3-connected for $i = 1, 2$, then G^* obtained by operation (1) or (2) from G also contains two nonincident edges f_1, f_2 s.t. $G^* \cdot f_i$ is 3-connected for $i = 1, 2$.

Proof of the claim. It is clear that $G^* \cdot e_i$ is 3-connected if G^* is obtained by operation (1). Assume G^* is obtained by operation (2) by splitting vertex v into v_1 and v_2. Since e_1 and e_2 are not incident in G, w.l.o.g v is not an endpoint of e_1 in G. It is clear that $G^* \cdot v_1v_2 \cong G$ is 3-connected.

Show that $G^* \cdot e_1$ is also 3-connected. Suppose not. Then since by the fact we have G^* is 3-connected, the minimum vertex cut of $G^* \cdot e_1$ must be $\{v_{e_1}, u\}$ for some u in $G^* \cdot e_1$.

If $u \neq v_i$, $i = 1, 2$, then $(G^* \cdot e_1 - \{v_{e_1}, u\}) \cdot v_1v_2 \cong (G^* \cdot e_1) \cdot v_1v_2 - \{v_{e_1}, u\} \cong G^* \cdot e_1 - \{v_{e_1}, u\}$ is not connected, contradicting with 3-connectivity of $G \cdot e_1$.

Now, w.l.o.g. we can assume $u = v_1$. Then $G^* \cdot e_1 - \{v_{e_1}, v_1\} \cong G \cdot e_1 - \{v_{e_1}\} - E_v$ is disconnected, where E_v is a collection of some edges incident to v in G. Since $G \cdot e_1 - \{v_{e_1}\} - E_v$ is disconnected, $G \cdot e_1 - \{v_{e_1}, v\}$ is also disconnected, contradicting with the 3-connectivity of $G \cdot e_1$. Thus we have $G^* \cdot e_1$ and $G^* \cdot v_1v_2$ are 3-connected.

It is clear that W_n, $n \geq 4$ contains two nonincident edges e_1 and e_2 s.t. $W_n \cdot e_i$ is 3-connected for $i = 1, 2$. Hence by the fact and the claim we have the result.

□

Graph Theory (I), Homework 2-6 by 張澍仁

Consider a minimal set **S** s.t. $\left\|[S,\bar{S}]\right\| = k$.

Case 1: |S| =1

=> Let S={v} , then d(v) = k

又 $\delta(G) \ge \kappa'(G) = k$

=> $\delta(G) = k$ □

Case 2: $|S| \neq 1$

1) claim: G[S] connected

Pf: If G[S] is not connected

Let $S_1 \subset S$, G[S_1] is a component of G[S]

Then $\rightarrow\leftarrow$ to S : minimal set. □

2) $\therefore E(G[S]) \neq \varnothing$

Take $G' = G - \{e\}$, e = xy $\in$ E(G[S])

=> G': k' edge connected , $k' < k$

=> $\exists T$, $\left\|[T,\bar{T}]\right\| = k' \le k-1$ in G'.

3) claim:

(a) $\left\|[T,\bar{T}]\right\| = k-1$ in G'

(b) $\left\|[T,\bar{T}]\right\| = k$ in G

(c) $S \cap T \neq \varnothing$

(d) $S \cup T \neq V(G)$

Pf:

(a) if $\left\|\left[T,\overline{T}\right]\right\|$ < k-1 in $G' = G - \{e\}$

Then $\left\|\left[T,\overline{T}\right]\right\|$ < k in G, a contradiction to G is k edge connected .

(b) if $\{x,y\} \in T$ (similar for $\{x,y\} \notin T \Rightarrow \{x,y\} \notin \overline{T}$)

Then T is also a set s.t. $\left\|\left[T,\overline{T}\right]\right\| = k-1$ in G , $\rightarrow\!\times\!\leftarrow$

$\therefore$ w.l.o.g. Let $\{x\} \in T$ and $\{y\} \notin T$.

$\because \left\|\left[T,\overline{T}\right]\right\| = k-1$ in G'

$\therefore \left\|\left[T,\overline{T}\right]\right\| = k$ in G □

(c) by (b)

$\therefore S \cap T \supseteq \{x\} \neq \varnothing$ □

(d) if $S \cup T = V(G)$

Then $T = V(G) - S'$, $S' \subset S$

$\Rightarrow \overline{T} = S'$, $|\overline{T}| = |S'| < |S|$

又 $\left\|\left[\overline{T},T\right]\right\| = \left\|\left[T,\overline{T}\right]\right\| = k$ in G

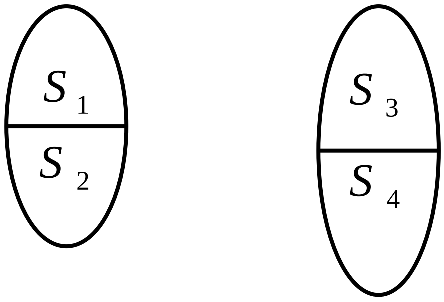
$\rightarrow\leftarrow$ to S : minimal. □

4) claim: For $S \subseteq V(G)$, let $d(s) = \left\|\left[S,\overline{S}\right]\right\|$

Let X,Y be nonempty proper vertex subsets of G.

Then $d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y)$

Pf: Let G:



$$S_1 \cup S_2 \cup S_3 \cup S_4 = V(G)$$

$$S_1 \cup S_2 = X \qquad S_1 \cup S_3 = Y$$

$$S_1 = X \cap Y \qquad S_1 \cup S_2 \cup S_3 = X \cup Y$$

$$\Rightarrow \quad d(X \cap Y) = \|[S_1, S_2]\| + \|[S_1, S_3]\| + \|[S_1, S_4]\|$$

$$d(X \cup Y) = \|[S_4, S_1]\| + \|[S_4, S_2]\| + \|[S_4, S_3]\|$$

$$d(X) = \|[S_1, S_3]\| + \|[S_1, S_4]\| + \|[S_2, S_3]\| + \|[S_2, S_4]\|$$

$$d(Y) = \|[S_1, S_2]\| + \|[S_1, S_4]\| + \|[S_2, S_3]\| + \|[S_3, S_4]\|$$

$$\Rightarrow \quad d(X) + d(Y) - d(X \cap Y) - d(X \cup Y) = 2\|[S_2, S_3]\| \geq 0$$

$\square$

5) By 前

$$2k < d(S \cap T) + d(S \cup T)$$
$$\leq d(S) + d(T) = 2k \qquad \rightarrow\leftarrow$$

$\therefore$ Q.E.D. $\square$

ps: $\because \varnothing \neq (S \cap T) \subset S \quad \therefore d(S \cap T) > k$

$\because (S \cup T) \neq V(G) \quad \therefore d(S \cup T) \geq k$

Graph Theory (I), Homework 2-7 by 黃皓文

Theorem (Mader[1978]): If z is a vertex of a multigraph G s.t. $d_G(z) \notin \{0,1,3\}$ and z is incident to no cut-edge, then z has neighbors x and y s.t. $\kappa'_{G-xz-yz+xy}(u,v) = \kappa'(u,v)$ for all $u,v \in V(G)-\{z\}$

Proof: By induction on $|G|$

Basic case:($|G|=2$).Let $V(G)=\{a,b\}$. If G is 2k-connected graph, then there are 2k multi-edge between a and b. We can orientate k edges form $a \to b$ and others k edges form $b \to a$.

In general case, let G be 2k-connected graph. There is a spanning subgraph G_0 of G is minimally 2k-connected. By EX6, there is a vertex z of G s.t. $d_{G_0}(z) = \delta(G_0) = 2$k.

Now, apply k time Mader Theorem to the vertex z,i.e. there is a multiset $N = \{(x_1, y_1),(x_2, y_2),...,(x_k, y_k)\}$ and let $G_i = G_{i-1} - x_i z - y_i z + x_i y_i$ for all i=1,2,…,k.

Note that $d_{G_k}(z) = 0$ we obtain a new graph $H = G_k - z$ with $V(H)=V(G)-\{z\}$ and

$\kappa'(H) = \min_{u,v \in v(H)} \kappa'_H(u,v) = \min_{u,v\in V(G_0)-\{z\}} \kappa'_{G_0}(u,v) \geq \kappa'(G_0)$ =2k. i.e. H is 2k-connected graph with $|H|=|G|$-1. By induction hypothesis, H has k-connected graph orientation D'.

Now, determine an orientation D_0 of G_0 from D' by following steps.

(1) if $x_i \to y_i$ (or $y_i \to x_i$) in D', then set $x_i \to z$ and $z \to y_i$ (or $y_i \to z$ & $z \to x_i$) in D_0.

(2) Copy other $u \to v$ in D' to D_0.

Then D_0 is a k-connected orientation of G_0. Since $G \supseteq G_0$, G also has k-connected orientation.

Graph Theory (I), Homework 2-8 by 張雁婷

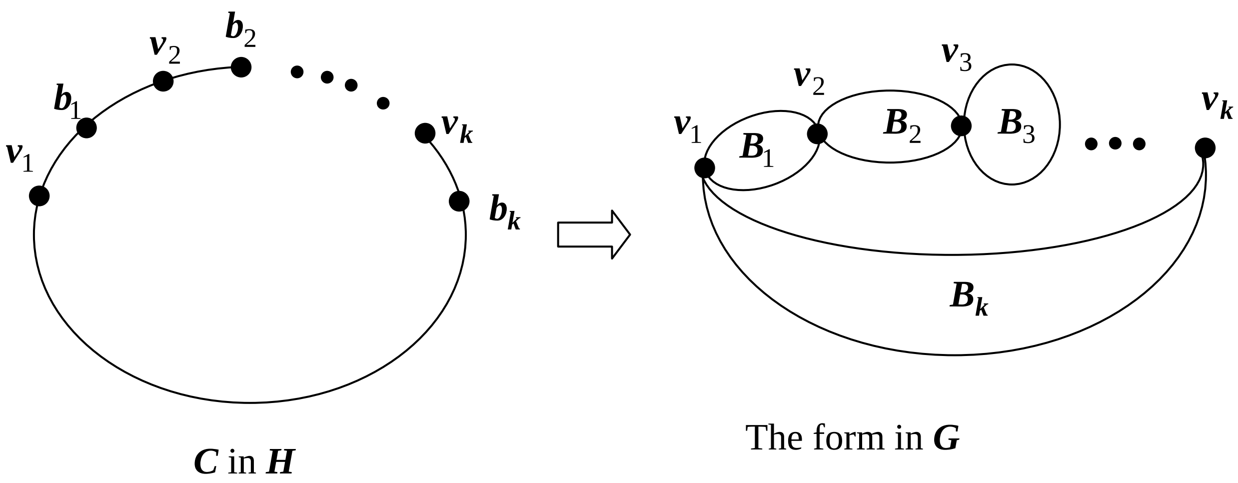
8. (a)

Let $H = V \cup B$ be a bipartite graph, where V is the set of the cut-vertices of G, and B is the set of vertices which represent the blocks of G.

Claim: H has no cycles, and hence H is a forest.

Pf:

Suppose H has a cycle C. Since H is bipartite, $|C|$ is even, and the vertices from V and B alternate with each other.



$\because B_i$ is the maximal connected subgraph of G that has no cut-vertex for each i.

$\therefore B_1 \cup B_2 \cup ... \cup B_k \cup \{V_1\} \cup \{V_2\} \cup ... \cup \{V_k\}$ induces a subgraph with no cut-vertex.

$\Rightarrow$ It contradicts that B_i is maximal.

$\therefore H$ has no cycle, and hence H is a forest. ■

(b)

Let's consider the block-cutpoint graph $\boldsymbol{H}$ of $\boldsymbol{G}$.

$\because \forall$ vertex $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{v}$ is a cut-vertex of $\boldsymbol{G}$.

$\therefore \deg_{\boldsymbol{H}}(\boldsymbol{v}) \geq 2, \forall \boldsymbol{v} \in \boldsymbol{V}$

And $\boldsymbol{H}$ is a forest, so there are at least 2 vertices with degree 1 in $\boldsymbol{H}$. From above, these two vertices are from $\boldsymbol{B}$, and their neighbors are both from $\boldsymbol{V}$.

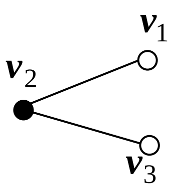
$\therefore$ Each of the two blocks which represented by the two vertices with degree 1 in $\boldsymbol{H}$ has exactly one cut-vertex in itself. ■

(c)

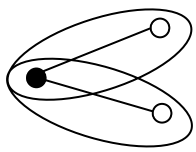
Use strong induction on $|\boldsymbol{V(G)}|$.

1° $|\boldsymbol{V(G)}| = 1$ or 2, $\boldsymbol{G}$ has no cut-vertex.

2° $|\boldsymbol{V(G)}| = 3$, there are only one case that $\boldsymbol{G}$ has a cut-vertex:



has two blocks :



$$1 + (\boldsymbol{b}(\boldsymbol{v}_1) - 1) + (\boldsymbol{b}(\boldsymbol{v}_2) - 1) + (\boldsymbol{b}(\boldsymbol{v}_3) - 1) = 2$$

3° Consider $|\boldsymbol{V(G)}| = \boldsymbol{n}$, and $\boldsymbol{G}$ has a cut-vertex and $\boldsymbol{c(G)} = \boldsymbol{k}$ (the number of components) by (b), we know that $\boldsymbol{G}$ has at least two blocks, called $\boldsymbol{B}_1, \boldsymbol{B}_2$, each of which contains exactly one cut-vertex of $\boldsymbol{G}$. Suppose the cut-vertex contained in $\boldsymbol{B}_1$ is $\boldsymbol{u}_1$. Let $\boldsymbol{G}'$ be the graph induced by $\boldsymbol{V(G)} - \boldsymbol{V(B_1)} + \{\boldsymbol{u}_1\}$ in $\boldsymbol{G}$. Then $|\boldsymbol{V(G')}| = |\boldsymbol{V(G)} - \boldsymbol{V(B_1)} + \{\boldsymbol{u}_1\}| < |\boldsymbol{V(G)}| = \boldsymbol{n}$, and we don't delete any cut-vertex from $\boldsymbol{G}$, so $\boldsymbol{c(G')} = \boldsymbol{c(G)} = \boldsymbol{k}$. Let $\boldsymbol{b}_{\boldsymbol{G}'}(\boldsymbol{v})$ be the number of blocks containing $\boldsymbol{v}$ in graph $\boldsymbol{G}'$, then by the induction hypothesis, $\boldsymbol{G}'$ has exactly

$\boldsymbol{k}+\sum_{v\in V(G')}(\boldsymbol{b}(v)-1)$ blocks. Therefore, G has exactly $\boldsymbol{k}+(\sum_{v\in V(G')}(\boldsymbol{b}(v)-1))+1$ blocks. And

$$\boldsymbol{k}+(\sum_{v\in V(G')}(\boldsymbol{b}(v)-1))+1$$

$$=\boldsymbol{k}+(b_{G'}(u_1)-1)+(\sum_{\substack{v\in V(G')\\ v\neq u_1}}(\boldsymbol{b}(v)-1))+1 \quad (\text{除了 } u_1 \text{ 外}, \ b_G(v)=b_{G'}(v) \ \text{ for } \ v\in V(G') \ )$$

$$=\boldsymbol{k}+((b_{G'}(u_1)+1)-1)+(\sum_{\substack{v\in V(G')\\ v\neq u_1}}(\boldsymbol{b}(v)-1))$$

$$=\boldsymbol{k}+(b_G(u_1)-1)+(\sum_{\substack{v\in V(G')\\ v\neq u_1}}(\boldsymbol{b}(v)-1))$$

$$=\boldsymbol{k}+\sum_{v\in V(G)-V(B_1)+\{u_1\}}(\boldsymbol{b}(v)-1)$$

$$=\boldsymbol{k}+\sum_{v\in V(G)}(\boldsymbol{b}(v)-1) \quad (\text{ Since } \ \forall x\in V(B_1)-\{u_1\}, b_G(x)=1 \therefore b(x)-1=0) \ \blacksquare$$

(d)

Let G have j cut-vertices, and $C=\{c_1,c_2,\ldots,c_j\}$ be the set of cut-vertices. For each cut-vertex c_i, let b_i be the number of blocks containing c_i. Note that $b_i\geq 2, \forall i\in\{1,\ldots,j\}$. From (c), we have known that a graph G with k components has exactly $k+\sum_{v\in V(G)}(b(v)-1)$ blocks.

And $k+\sum_{v\in V(G)}(b(v)-1)= \ k+\sum_{i=1,2,\ldots j}(b_i-1)+\sum_{v\in V(G)-C}(b(v)-1)$

$$=k+(\sum_{i=1}^{j}b_i-j)+0$$

$$\geq k+\sum_{i=1}^{j}2-j=k+2j-j=k+j>j \quad \text{since} \quad k>1$$

Hence, the number of cut-vertices is less than the number of blocks. ■

Graph Theory (I), Homework 2-9 by 曾雅榕

Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \geq n(G) - 2$. Prove that this is best possible for each $n \geq 4$ by constructing a simple n-vertex graph with minimum degree $n-3$ and connectivity less than $n-3$

Ex4.1.19(D.B.West)

Proof:

(1) Let $n(G) = n$

1. consider $\delta(G) = n-1$

   Then $G = K_n$. $\kappa(G) = n-1 = \delta(G)$ O.K

2. consider $\delta(G) = n-2$

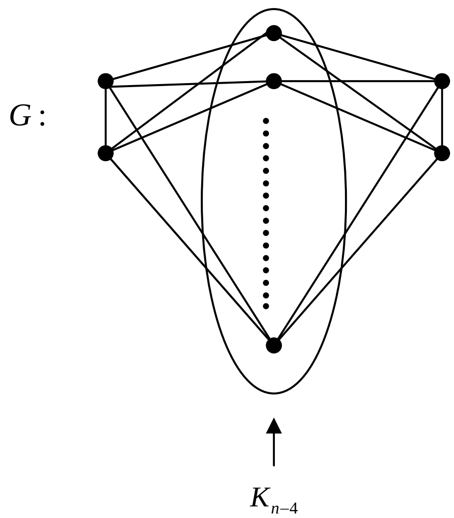
   Suppose u is nonadjacent to v in G.

   Then the other $n-2$ vertices are common neighbor of u and v.

   To separate u and v, we must delete the $n-2$ vertices.

   Hence $\kappa(G) = \delta(G) = n-2$

(2)



Then $\delta(G) = n-3$ but $\kappa(G) = n-4 < n-3$

Hence if $n \geq 4$,we can construct a graph with $\delta(G) = n-3, \kappa(G) = n-4$.

This proves that above is the best case for $\delta \geq n-2$.

Graph Theory(I), Homework 2-10 by 羅元勳

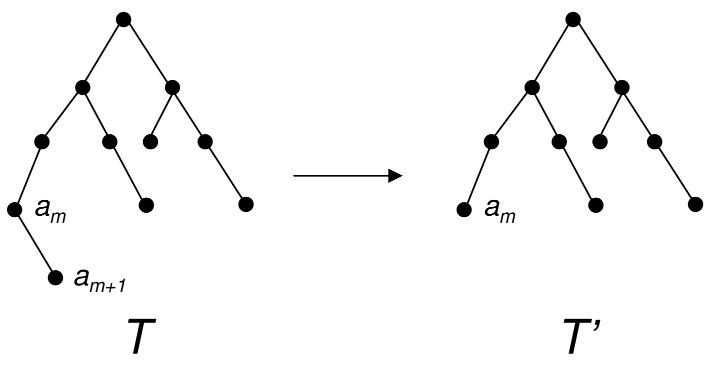
Let G be a $2m$-regular graph, and let T be a tree with m edges. Prove that if the diameter of T is less than the girth of G, then G decomposes into copies of T.

Proof. Let $V(T) = \{a_1, a_2, \cdots, a_{m+1}\}$. It suffices to prove that G has a decomposition where each vertex v in G appears in $m+1$ copies of T such that v corresponds to $a_1, a_2, \cdots, a_{m+1}$ exactly once.
By induction on m.

1. $m = 1$. G is 2-regular and $T = P_2$.
   It's trivial that G decomposes into copies of T.

2. Suppose that the assertion is true for $m - 1$.

3. Let G be a $2m$-regular graph.
   By Theorem 3.3.9. G has a 2-factor, denotes as H. Consider that $G\,' = G - H$ is $(2m-2)$-regular and girth $g(G\,') \geq g(G)$. Since T is a tree, W.L.O.G., let $a_{m+1} \in V(T)$ s.t. $deg(a_{m+1}) = 1$. Note that a_{m+1} is adjacent a_m in T and let $T' = T \setminus \{a_{m+1}\}$. Thus, T' is a tree with $diam(T') \leq diam(T) < g(T) \leq g(G\,')$.

   By induction hypothesis, $G\,'$ has a decomposition where each vertex v in $G\,'$ appears in m copies of T' such that v corresponds to $V(T')$ exactly once.

Now, we want to add back $E(H)$ to G' such that each copy of T' receives one edge to become T.

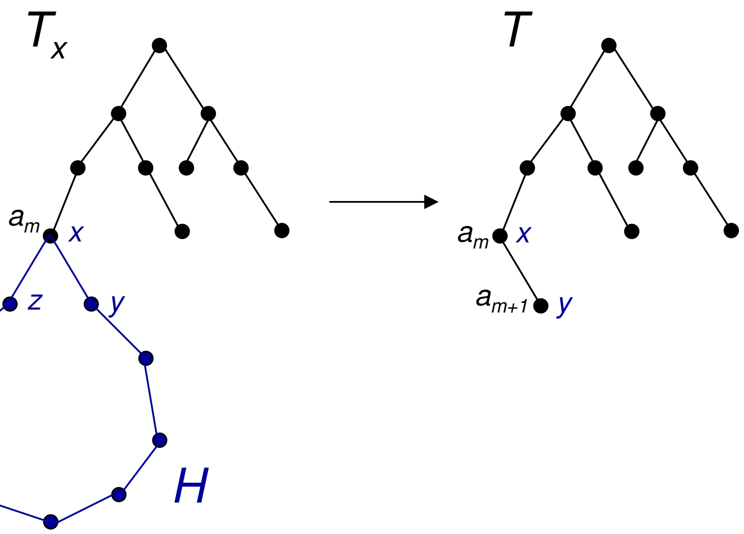
For each $x \in V(G)$, there is a copy of T', T_x contains x such that x corresponds to a_m in T_x.

Let $N_H(x) = \{y, z\}$.

Claim: $y, z \notin V(T_x)$**.**

W.L.O.G., suppose $x \in V(T_x)$. Since there is a path P connects x and y in T_x, and x is adjacent to y in G. Then $a - P - b - a$ is a cycle with length at most $diam(T)$. But $diam(T) < g(G)$. $\rightarrow\leftarrow$

Hence we can add the edge xy to T_x to induce T as following.



For each $x \in V(H)$, we find out the corresponding T_x and add the next vertex of x in H to T_x. Then we can partition G into T's.

■