Graph Theory (I), Homework 1-1　by 蔡佩純

Q1: Let a rectangle T be tiled with rectangles $T_1, T_2\ldots T_l$. Prove that if each T_i has an integral side, then T also has an integral side.

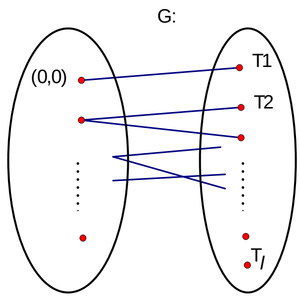
A1:
(方法一)

1^o 將 T 放在第一象限的格子點上(Z_2),且左下角落於(0,0) ,右上角的點爲(s,t).

2^oDefine G, G is bipartite ,and A、B are bipartite sets of G ,

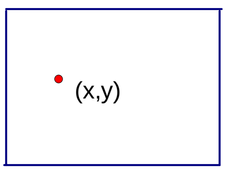
where $A = Z^2 \cap T, B = \{T_1, T_2, \cdots, T_l\}$

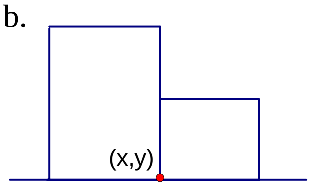
and (x,y) ~T_i if (x,y)在 T_i 的頂點上.

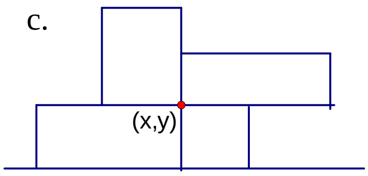


3^o(1)$\because$ (0,0)只在 T_1 的頂點上$\Rightarrow$ deg(0,0) = 1

(2)Consider (x,y)不在 T 的頂點:

a. 

b. 

c. 

deg((x,y))=0　　deg((x,y))=2　　deg((x,y))=4

$\therefore$ $\deg((x,y)) \in \{0,2,4\}$ for (x,y)不爲 T 的頂點.

(3)Consider T_i 的四個頂點:

Since T_i has an integral side$\Rightarrow$ $\deg(T_i) \in \{0,2,4\}$.

4^o Since $\sum_{v\in V(A)} \deg(v) = \sum_{T_i \in V(B)} \deg(T_i)$ is even, and by (1),(2),(3)

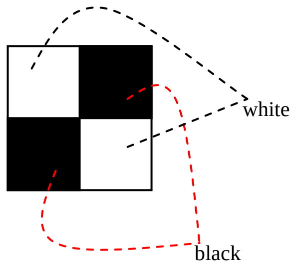
$\Rightarrow$A 中的另一個 odd degree 必出現在 T 的頂點中.(除了(0,0)外的 3 點).

Hence T has an integral side.

(方法二)

1^o 將 T 放在第一象限的格子點上(Z_2),且左下角落於(0,0) ,右上角的點爲(s,t).

2^o 將每個 1x1 的格子點分割成四等分，並塗成如右:



3^oSince $\forall T_i$, T_i has an integral side.

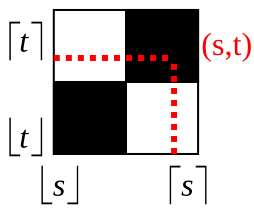
$\Rightarrow T_i$ has the same area of black and white.

$\Rightarrow$ T has the same area of black and white.

4^oIf s and t are both not integers, then $\lfloor s \rfloor + 1 = \lceil s \rceil, \lfloor t \rfloor + 1 = \lceil t \rceil$.

Both $[0,\lfloor s \rfloor]\times[0,t]$ and $[\lfloor s \rfloor, s]\times[0,\lfloor t \rfloor]$ have the same area of black and white.

The square $[\lfloor s \rfloor,\lceil s \rceil]\times[\lfloor t \rfloor,\lceil t \rceil]$ must have the same area of black and white.



But , since s and t are not integers.

$\therefore$ (s,t)不論在哪個區域均不能平分黑與白的面積.

Hence one of s or t is an integer. i.e. T has an integral side.

Graph Theory (I), Homework 1-2 by 黃皓文

Lemma: If *G* is a graph with $V(G)=\{v_1,v_2,...,v_p\}$, $p\geq 3$, then *G* is connected if and only if at least two of the subgraphs $G_i=G-v_i$ are connected.

Proof:

Observe that if *G* contains an isolated vertex, then *G* is reconstructible.(Degree seq. is reconstructible).
Assume that *G* contains no isolated vertices. Let *F* is a component of maximum order of G, then *F* is a component of maximum order among the components of $G_i=G-v_i$, $i=1,2,...,p$. So *F* is recognizable. Now, delete a vertex that is not a cut-vertex from F, obtaining F'. Let G has *n* components which are isomorphic to F.

Find $S=\{G_j \mid c(G_j)=c(G) \text{ and } G_j \text{ has a minimum } m \text{ of components isomorphic to F}\}$. It is clear that *n*=*m*+1. Next, find

$S'=\{G_j \mid G_j \text{ has a maximum number of components isomorphic to } F'\}$.

Then pick some $G_i=G-v_i\in S'$.And G can be reconstructed from G_i by replacing a component of G_i isomorphic to F' by a component isomorphic to F.

Graph Theory (I), Homework 1-3 by 陳柏澍

3. Use adjacency matriz of G to find the number of $5-cycles$ in G.

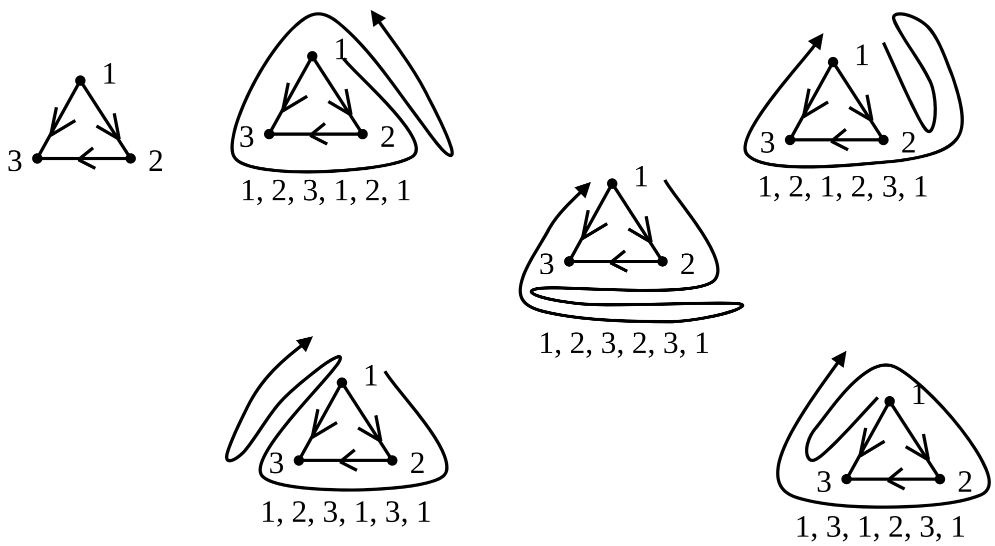
Let $A=(a_{ij})$ be the adjacency matrix of G.

Note. # of closed $5-walks$ is $tr(A^5)$.

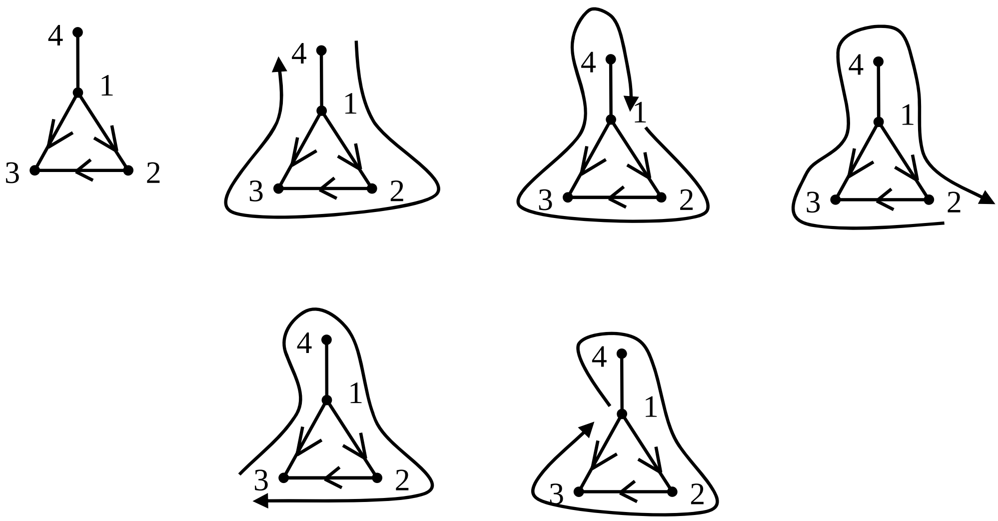
# of closed $3-walks$ is $tr(A^3)$.

# of distinct $3-walks$ from v_i to v_i is a_{ii}.

A closed $3-walk$, say 123, will generate 5 closed $5-walks$.



A closed $3-walk$, say 123, together with an another edge will generate 5 closed $5-walks$.



$$\#\text{ of } 5-cycles = \frac{1}{10}(\#\text{ of } closed\text{ 5-walks } - \#\text{ of } closed\text{ 5-walks with 3 } edges -$$

$$\#\text{ of } closed\text{ 5-walks with 4 } edges)$$

$$= \frac{1}{10}(tr(A^5) - 5tr(A^3) - 5\sum_{i=1}^{V(G)}(\sum_{j=1}^{V(G)} a_{ij} - 2)a_{ii}^3).$$

Graph Theory (I), Homework 1-4 by 張澍仁

Theorem : (Tutte's theorem)

A graph G has a 1-factor

<=> o (G-S) ≦ |S| , for all S⊆V(G) .

Let G : bridgeless cubic graph .

Claim : o (G-S) ≦ |S| , for all S⊆V(G)

Pf : Let H be a odd component of G-S.

Let m be the number of edges from S to H.

$$\because \sum_{v \in V(H)} d_H(v) = 3 * |V(H)| - m$$

where $\sum_{v \in V(H)} d_H(v)$ is even and $|V(H)|$ is odd.

∴ m : odd.

又 G : bridgeless , ∴ m≧3

=> 3*o (G-S)≦ G 中所有 odd components 連到 S 的邊數

= S 連到所有 odd components 的邊數

≦ S 最大可能的邊數

= 3*|S|

=> o (G-S) ≦ |S| , for all S⊆V(G) . □

Then by the Claim , G has a 1-factor.

∴Q.E.D.

Graph Theory (I), Homework 1-5 by 張雁婷

Prove that for each $g \geq 3$, a $(3,g)$-cage has no bridge.

Pf.

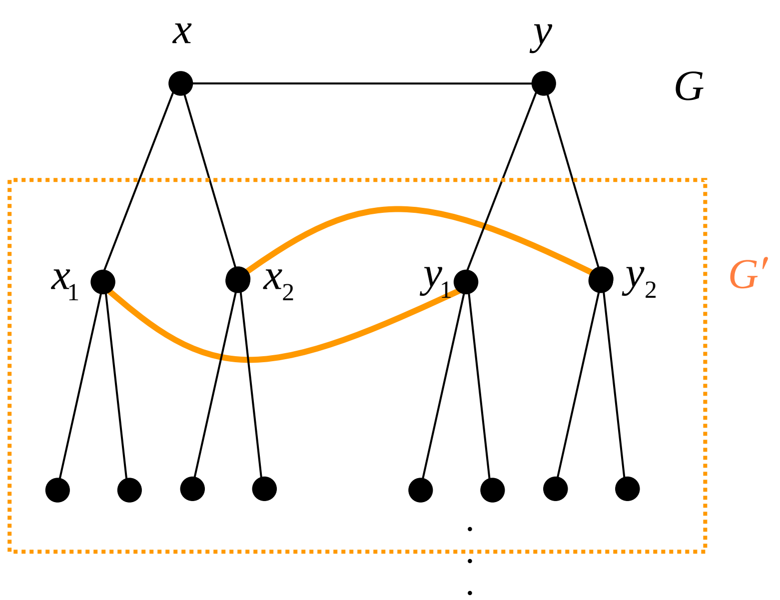
Assume we have known the lemma below: (skipping the proof)

Lemma $\forall k \geq 3$ and $3 \leq g_1 < g_2 \Rightarrow f(k;g_1) < f(k;g_2)$, where

$f(k;g)$ is the number of vertices of a (k,g)-cage.

Suppose there is a bridge $e(x,y)$ in a $(3,g)$-cage G. Then $G-e(x,y)$ has 2 components, called G_1 and G_2. W.L.O.G., let $x \in V(G_1)$ and $y \in V(G_2)$. Then $G_1 - x$ has two vertices x_1, x_2 with degree 2, and $G_2 - y$ also has y_1, y_2 with degree 2. Then we add $e(x_1,y_1)$ and $e(x_2,y_2)$ to connect $G_1 - x$ and $G_2 - y$, and obtain a new graph G'. Then G' is 3-regular and $g(G') \geq g(G)$, but $|V(G')| = |V(G)| - 2 < |V(G)|$, a contradiction.

Hence, a $(3,g)$-cage has no bridge! ■



Graph Theory(I), Homework 1-6 by 羅元勳

Prove that if $k \geq 2$ and $k \left| \binom{n}{2} \right.$, then the edge set of K_n can be partitioned into k sets $E_1, E_2, \cdots, E_k$ such that they induce k isomorphic subgraphs of K_n.

Proof.

1. **(Tarsi's Theorem.)** If $k \left| \binom{n}{2} \right.$ and $k \geq \left\lceil \frac{n}{2} \right\rceil$, then $E(K_n)$ can be partitioned into k paths with length $\frac{n(n-1)}{2k}$.

2. $E(K_{2k})$ can be partitioned into k Hamiltonian paths, and each vertex be an endpoint in a path exactly once.

Pf. Construct by deleting a vertex from the decomposition of K_{2k+1} into k Hamiltonian cycles.

3. Let $n = 2k \cdot m + a$ where $a < 2k$.
   Then partition K_n into $G_1 \cup G_2 \cup \cdots \cup G_m \cup G_{m+1}$ where $V(G_i) = \{v_{i,j} | \; j = 1, 2, \cdots, 2k\}$, for $i = 1, 2, \cdots, m$.
   By 1. and 2., for each i, G_i can be partitioned into k paths.
   For convenient, let $\{P_{i,1}, P_{i,2}, \cdots, P_{i,k}\}$ be the path partition of G_i, $1 \leq i \leq m+1$. And for $1 \leq i \leq m$, $1 \leq j \leq k$, let $\{v_{i,2j-1}, v_{i,2j}\}$ be the endpoints of $P_{i,j}$.

   Define $H_j = \bigcup_{i=1}^{m+1} P_{i,j} \; \cup \bigcup_{i=1}^{m} \bigcup_{q=1}^{2k} \bigcup_{p=i+1}^{m+1} \{v_{i,2j-1}v_{p,q}, v_{i,2j}v_{p,q}\}$.

   Then $\bigcup_{j=1}^{k} E(H_j) = E(K_n)$, and $E(H_i) \cap E(H_j) = \phi$. Hence we have done.

■

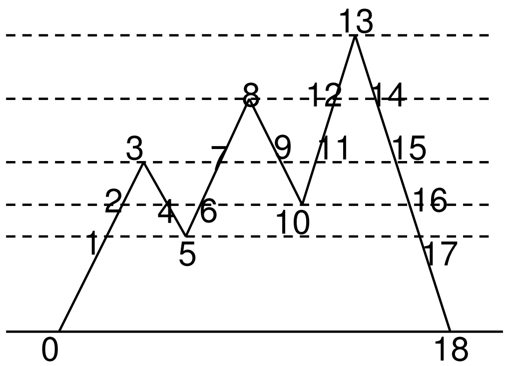
Graph Theory(I), Homework 1-7 by 陳宏賓

PROBLEM.

Hikers A and B begin at $(a, 0)$ and $(b, 0)$, respectively. Prove that A and B can meet by travelling on the mountain range in such a way that at all times their heights are the same. (Hint: Define a graph to model the movements.) Ex. 1.3.13 (D. B. West)

Proof.

在每個山峰及山谷畫延伸的水平線, 一旦水平線碰到山陵線就定義一個新的點, 並由 A 點到 B 點分別將這些點標號:$\{0, 1, \cdots, n\}$. Then, denote order pair (a, b) as A in a and B in b at a certain time. (see the following figure)



Define a graph $G = (V, E)$, where $V = \{(a, b) : 0 \leq a \leq b \leq n, \quad a, b \in \mathbb{Z}\}$ and $(a_1, b_1) \sim (a_2, b_2)$ if and only if A,B 可同時分別由a_1, b_1經過一步走到a_2, b_2. Then, it suffices to show that there is a path from $(0, n)$ to $(t, t) \in V(G)$ for some t.

Observe that for any two distinct a, b with $(a, b) \neq (0, n)$, we have $deg(a, b) \in \{0, 2, 4\}$. Thus, only those vertices (t, t) can have an odd degree. Since it must be even that the total sum of vertex degrees in any component of G, there is an odd degree vertex in the same component with $(0, n)$, i.e., there is a path from $(0, n)$ to the odd degree vertex in G. Hence there exists a vertex (t, t) for some t such that A meet B at the vertex.

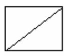
Graph Theory(I), Homework 1-8 by 吳政軒

8. G is a T-graph if every edge in G belongs to a △ , $\begin{cases} \frac{3n-3}{2} & ,\text{if } n \text{ is odd} \\ \frac{3n-2}{2} & ,\text{if } n \text{ is even} \end{cases}$ 即為所求

Pf: By induction on n

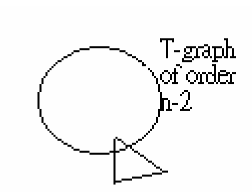
Basis case：

n=3 $\frac{3n-3}{2}=3$ => △ is the only T-graph of order 3

n=4 $\frac{3n-2}{2}=5$ =>  is the only T-graph of order 4

In general case,

If n is odd => n-2 is odd



is a graph of order n

⇨ the number of edges is $\frac{3(n-2)-3}{2}+3=\frac{3n-3}{2}$

Let G be a T-graph of order n such that $\|G\|<\frac{3n-3}{2}$

=> ∃ a vertex v_1 of degree=2 , Let $N(v_1)=\{v_2,v_3\}$, then v_2,v_3相鄰

Case(a)：

In $G-v_1$, v_2,v_3 has common neighbor ,then $G-v_1$ is also a T-graph

$\|G-v_1\| \geq \frac{3(n-1)-2}{2}=\frac{3n-5}{2}$ => $\|G\| \geq \frac{3n-5}{2}+2=\frac{3n-1}{2} \geq \frac{3n-3}{2}$

Case(b)：

In $G-v_1$, v_2,v_3 has no common neighbor ,

then $(G-v_1)□v_2v_3$ is also a T-graph

$\|(G-v_1)□v_2v_3\| \geq \frac{3(n-2)-3}{2}=\frac{3n-9}{2}$ => $\|G\| \geq \frac{3n-9}{2}+3=\frac{3n-3}{2}$

Hence $\frac{3n-3}{2}$ is optimal solution if n is odd.

Similar to n is even.

Graph Theory(I), Homework 1-9 by 余國安

9. Let $d_1 \le \cdots \le d_n$ be the vertex degrees of a simple graph G. Prove that G is connected if $d_j \ge j$ when $j \le n-1-d_n$. (Hint: Consider a component that omits some vertex of maximum degree.)

Proof. Suppose $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $deg_G(v_i) = d_i \quad \forall\, 1 \le i \le n$.

Suppose G is not connected. Let H be a connected component of G not containing v_n with degree sequence $d'_1 \le d'_2 \le \cdots \le d'_k$. Then $k \le n-1-d_n$, since G is simple. Thus $d'_k \ge d_k \ge k$, which is a contradiction. Hence G is connected. □

Graph Theory(I), Homework 1-10 by 張惠蘭

PROBLEM.

Let S be an n-element set, and let $\{A_1, \cdots, A_n\}$ be n distinct subsets of S. Prove that S has an element x such that $A_i \cup x$ are distinct for all i. Ex. 2.1.76 (D. B. West)

Proof.

Let S be an n-element set, and let $\{A_1, \cdots, A_n\}$ be n distinct subsets of S. In the following, view A_i as a vertex. Let G be a graph with vertex set $V = \{A_1, \cdots, A_n\}$. $A_iA_j \in E(G)$ iff A_i and A_j have exactly one different element.

Label edges of G by the different element of A_i and A_j if $A_iA_j \in E(G)$. Denote the label as $l : E(G) \to S$.

If $l(E(G)) \neq S$, then for any $x \in S - l(E(G))$, $A_1 \cap \{x\}, \cdots, A_n \cap \{x\}$ are all distinct. Suppose $l(E(G)) = S$. Then we can find a subgraph H of G s.t. $l(E(H)) = S$ and $l(e) \neq l(e')$ for any $e \neq e'$ in $E(H)$. It is clear that $|E(H)| = n$.

On the other hand, the number of edges in a cycle of G with the same label must be even, since the element deleting from A_i on the cycle must be added back along the cycle.

Thus by the choice of H we must have that H contains no cycle. So, $|E(H)| \leq n-1$, contradicting with the previous result. Hence we have the result.