GraphTheory(I), Homework(II)

Due Nor. 23, 12:00.

1. Prove Menger's Theorem in as many ways as possible.
2. Find a graph with (1) $\kappa(G) > l(G) > 0$ and (2) $\kappa(G) - l(G)$ is as large as possible.
3. Let $k \geq 2$. Show that every k-linked graph of order at least $4k$ contains a cycle of length at least $4k - 2$.
4. For each graph G of order p, prove that $\alpha_1(G) + \beta_1(G) = \alpha(G) + \beta(G) = p$.
5. Prove that there exist at least two edges e and f in a 3-connected graph $G \not\cong K_4$ such that G/e and G/f are 3-connected.
6. A k-edge-connected graph G is **minimally k-edge-connected** if for every $e \in E(G)$ the graph $G - e$ is not k-edge connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected. Ex. 4.2.37 (D. B. West)
7. Prove that every $2k$-edge-connected graph has a k-edge-connected orientation. Ex. 4.2.38 (D. B. West)
8. Let H be the block-cutpoint graph of a graph G that has a cut-vertex. Ex. 4.1.34 (D. B. West)
   (a) Prove that H is a forest.
   (b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.
   (c) Prove that a graph G with k components has exactly $k + \Sigma_{v \in V(G)}(b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.
   (d) Prove that every graph has fewer cut-vertices than blocks.
9. Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \geq n(G) - 2$. Prove that this is best possible for each $n \geq 4$ by constructing a simple n-vertex graph with minimum degree $n - 3$ and connectivity less than $n - 3$. Ex. 4.1.19 (D. B. West)
10. Let G be a $2m$-regular graph, and let T be a tree with m edges. Prove that if the diameter of T is less than the girth of G, then G decomposes into copies of T. Ex. 3.3.21 (D. B. West)